

Angle Between Vectors In The Plane

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# Introduction

Vectors have significant importance in vector geometry and physics. In particular, the direction of vectors and the angles at which they are oriented are critical in determining the effect that the combination of these vectors will have.

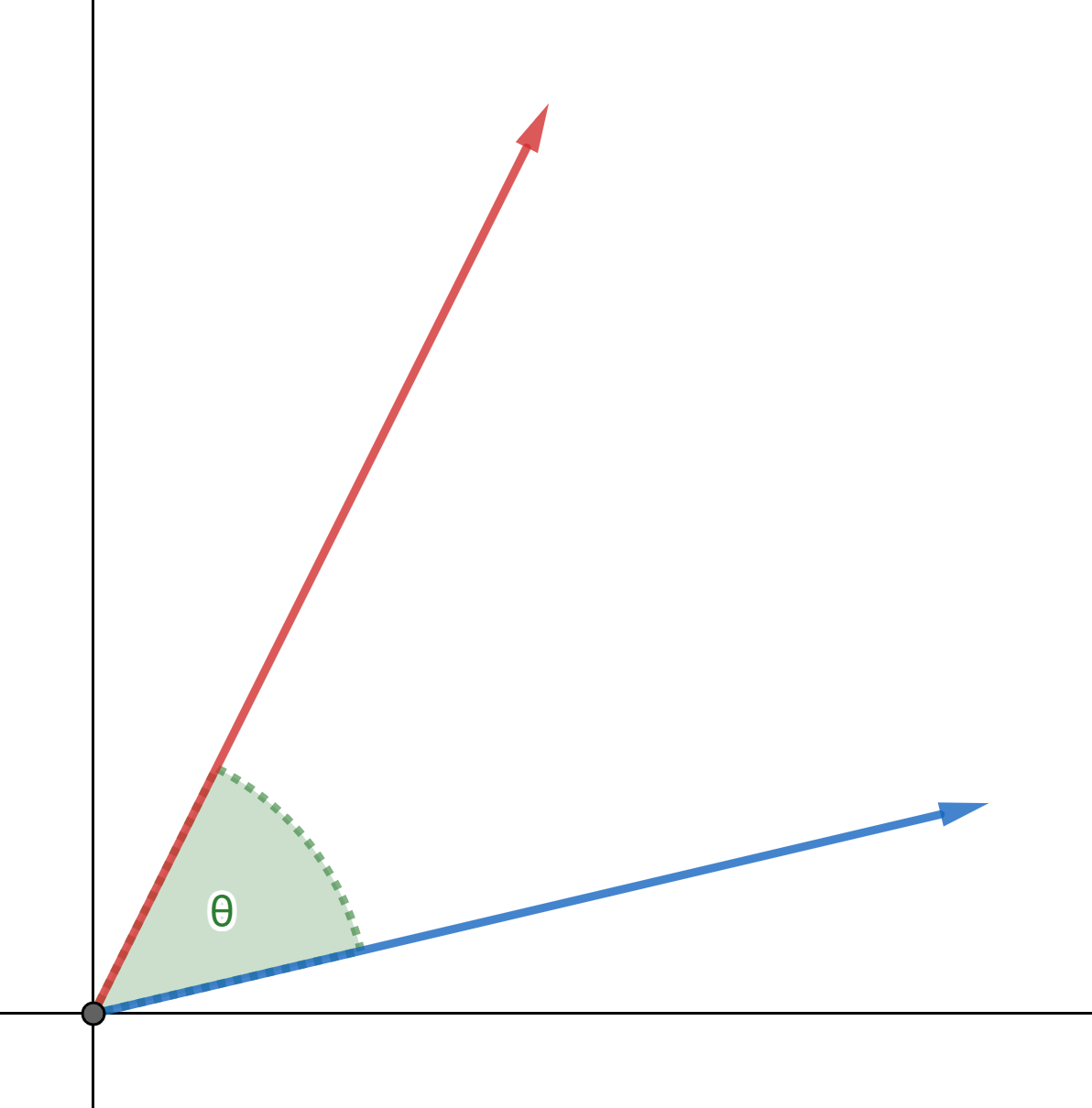
For example, if we study the movement of a soccer ball during a game, its position relative to the center of the field can be described by a position vector and the movement by a velocity vector whose length indicates the speed of the ball, so it will be the longer the faster the ball goes. The direction of the velocity vector explains the direction of the ball's movement.

Sometimes we are dealing with two vectors acting on the same object, so the angle of the vectors is critical. In the real world, any system is subject to several vectors combined together.

If there are two vectors in a plane such that the tails of both vectors are joined, then we can define the angle between them as:

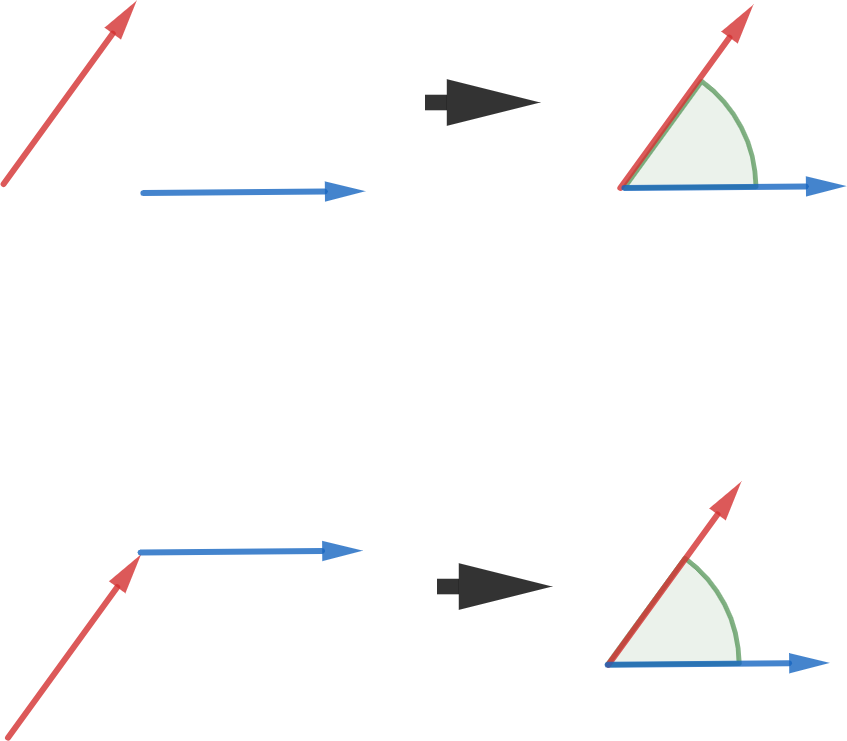
*“Angle between two vectors is the shortest angle at which any of the two vectors is rotated about the other vector such that both of the vectors have the same direction”.*

The discussion of vector angles focuses on finding the shortest angle between the vectors. This will focus on the angle between two vectors in standard position.  
  
*“A vector is said to be in standard position if its initial point is the origin (0, 0).”*

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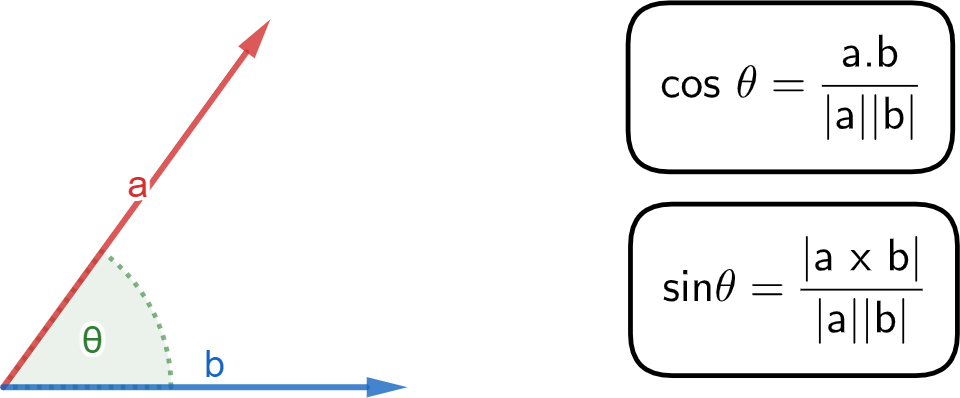
In other words, the angle between two vectors is the angle between their tails. Note that the angle between two vectors is always between 0° and 180°.

If the vectors are not joined tail-to-tail then we need to join them by shifting one of the vectors.



Through vector multiplication it is possible to find the angle between two vectors. To solve the vector multiplication two different methods can be used: scalar product and cross product.

Via the scalar product of two vectors a scalar quantity is obtained. On the other hand, as the name suggests, the vector product (or cross product) between two vectors produces a vector quantity.



# Angle Between Two Vectors Using Dot Product

Consider two vectors a and b separated by some angle θ. The formula of the dot product is:

where **a.b** is the dot product of two vectors. |a| and |b| are the magnitude of vectors **a** and **b,** and θ is the angle between them.  
The previous formula states that the dot product of two vectors a and b is equal to the product of their magnitudes multiplied by the cosine of the angle.  
So we start from the definition of the dot product to find the value of the angle.  
Let’s starts by isolating the cosine:

Finally, to find the angle between two vectors, a and b, we will solve the angle θ,

Let’s focus on the dot product, to this purpose consider two vectors **a** and **b**

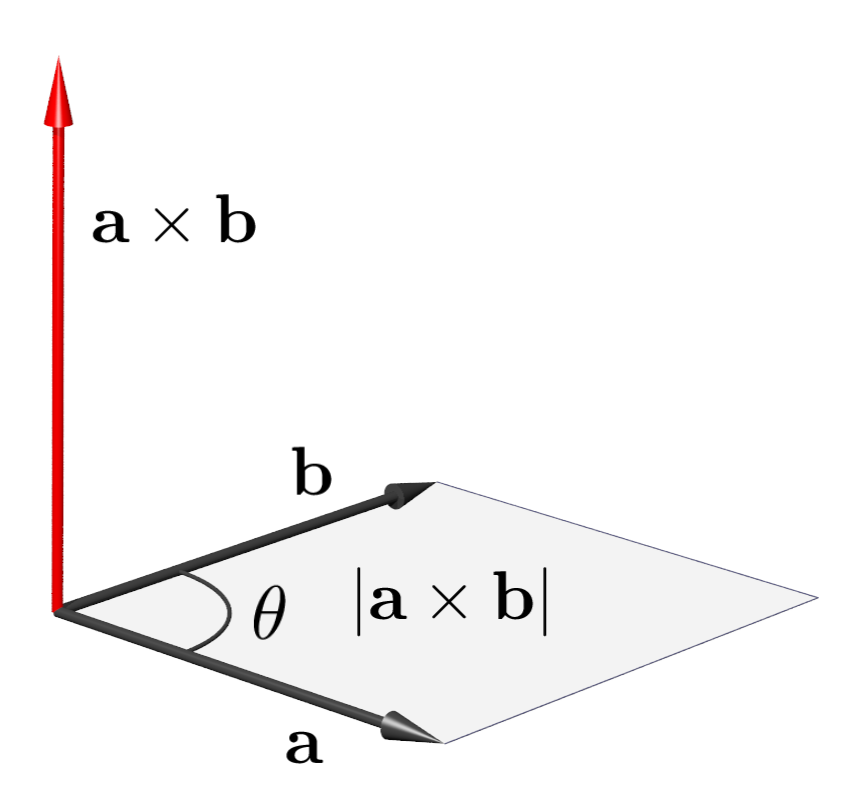
Then the dot product between two vectors **a** and **b** is given as:

**. = +**

# Angle Between Two Vectors Using Cross Product

Another method of finding the angle between two vectors is the cross product.  
Cross product is defined as:

*“The vector that is perpendicular to both the vectors and direction is given by the right-hand rule. “*

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The formula of the cross product is:

Where **θ** is the angle between two vectors, |a| and |b| are the magnitudes of two vectors **a** and **b,** and **n** is the unit vector perpendicular to the plane containing **a** and **b**. Its direction is given by the right-hand rule.

To solve this for θ, let us take magnitude of both members:

But since n is a unit vector, its magnitude is 1. So we get:

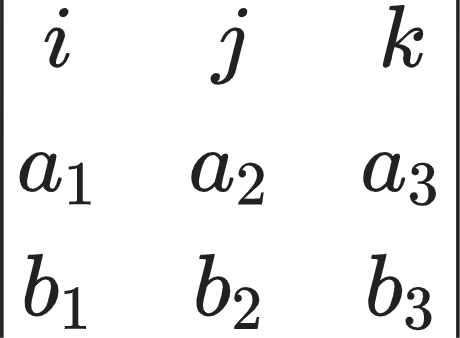
Let’s isolate the sinθ in order to find the angle between those vectors

Finally we can obtain the angle as:

Let’s focus on the cross product. Since we are going to use the cross product we have to consider the third dimension as well because the cross product will be a vector perpendicular to the plane containing a and b (so it won’t stay into their same plane).

So generally speaking we can take as example two tridimensional vectors a and b: such as

The cross product can be expressed as the determinant of the matrix



where is a positively oriented orthonormal basis.

Calculating the determinant we obtain

so we get the following vector

In our case study, since we are considering the angle between vectors in the xy plane, we can simplify the notation of **a** and **b** setting their third component to 0, this will make them two-dimensional vectors. Keep in mind that, as result of the cross products, we’ll still obtain a perpendicular vector, which will be perpendicular to the

xy plane containing **a** and **b**.

If we recalculate the previous formulas considering **a** and **b** as belonging to the xy plane (so with ), we obtain:

# Solved Problems

## Example 1

*Assignment:*

Find the angle between vectors **a** = <1, 2> and **b** = <-2, -1> using the **dot product**.

*Solution:*

Let θ be the angle between vectors a and b.

Let us find the angle θ between vectors using the dot product.

To use the formula we need to compute the dot product and magnitudes of both vectors.

Now we can calculate the angle as:

**143.13°**

## Example 2

*Assignment:*

Find the angle between vectors **a** = <1, 2> and **b** = <-2, -1> using the **cross product**.

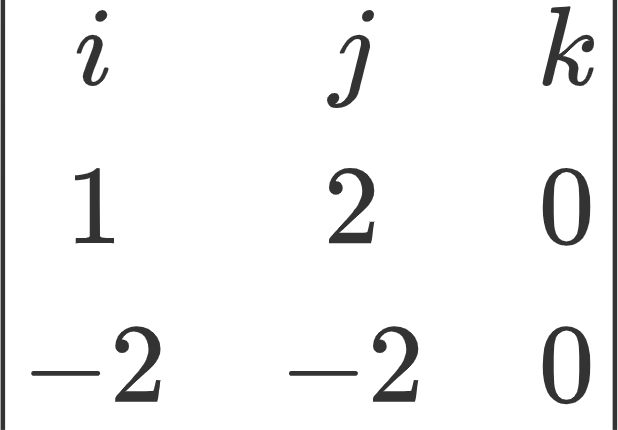
*Solution:*

Let θ be the angle between a and b. Let us find the angle θ between vectors using the cross product.

Since we are going to use the cross product we have to consider the third dimension, so we need to extend our vectors to the third dimension.

Update our notation of a and b:

Let us compute the cross product of **a** and **b**.



Now we find its magnitude.

We now want to use the formula to get the value of the angle

Then we get

We obtain θ ≈ **36.87 (or) 143.13°** (= 180 - 36.87) (as [sine](https://www.cuemath.com/trigonometry/sine-function/) is positive in the second quadrant as well).

## Example 3

*Assignment:*

Find the angle between vectors **a** = <0, 5> and **b** = <2, 0> using the **dot product**.

*Solution:*

Let θ be the angle between vectors a and b.

Let us find the angle θ between vectors using the dot product.

To use the formula we need to compute the dot product and magnitudes of both vectors.

Now we can calculate the angle as:

**90°**

*Notes*:

There are few considerations that can be done to speed up the reach of this solution.  
First of all the computation of and can be simplified since they are one-dimensional vectors (one of their components is 0) so their modulo is equal to their not zero component.

Moreover the computation of and , even if simple, is not necessary at all. Since we discovered that is equal to 0 and this will be the numerator of the arccos argument, it is unnecessary to evaluate the denominator as well.

But, going further, this exercise could be resolved without any calculations, but only with geometric considerations. Since **a** is a vertical vector (its x component is 0) and **b** is an horizontal vector (its y component is 0), we can infer that they are orthogonal vectors, this means that the angle between them is 90°.

# National Evaluation Exercise

(Maturity Examination - Italy:

<https://drive.google.com/file/d/16bxAx7d0ts5zgr3P62qzGu0ZPoZ2aywl/view?usp=sharing>)

PROBLEM 1

Consider triangles whose base is AB = 1 and whose vertex C varies so that the angle C Aˆ B is

keeps double the angle A Bˆ C .

1. Referring the plane to a convenient coordinate system, determine the equation of the locus γ geometric locus described by C.

2. Represent γ, taking into account, of course, the prescribed geometric conditions.

3. Determine the amplitude of the angle A Bˆ C that makes the sum of the squares of the heights relative to the sides AC and BC and, with the help of a calculator, give a value of it approximated in degrees and primes (sexagesimal).

4. Prove that A Bˆ C = 36° then AC=