Project funded by: Erasmus+ / Key Action 2 - Cooperation for innovation and the exchange of good practices, Strategic Partnerships for school education (European Commission, EACEA)

MATHESIS
CREATIVE MATH - VIRTUAL WORLD

Reports, Good Practices and Roadmaps in teaching mathematics at secondary and upper secondary level around Europe

MATHESIS – Attracting students' interest in mathematics and improving their Skills, comprehension and performance with the use of a virtual reality educational platform

2020-1-RO01-KA201-080410

September 2021

Project funded by: Erasmus+ / Key Action 2 - Cooperation for innovation and the exchange of good practices, Strategic Partnerships for school education (European Commission, EACEA)

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I. Background of Teaching Mathematics in Secondary Education around Europe

1. Theories based on the evolution of mathematics education

The Italian Leonardo Fibonacci (circa 1170 - 1250) is considered by many historians the most important mathematician of the Middle Ages.

It has gone down in history through at least two things: through its decisive contribution to the spread in Europe of the Hindu-Arabic numbering system, a fact achieved through its book Liber Abacci (The Book of the Abacus), as well as through the series that bears its name today. This recurring series, which appears in a problem in Liber Abacci, has fascinated and continues to fascinate mathematicians and non-mathematicians alike.

The information about his life is precarious - most of it is taken from the prefaces of the books he wrote. His date of birth is uncertain. His father, Guglielmo Fibonacci, was the envoy of the city of Pisa to defend the interests of the merchants of Pisa. Born in Italy, he spent most of his childhood in Bugia (today Bejaia, Algeria), where he learned with Arabic teachers the Arabic numbering system, as well as its use in practice.

At that time (12th-13th centuries), Pisa, Venice and Genoa were the great commercial centers of the Mediterranean, and their traders enjoyed traffic and trade facilities. Benefiting from these freedoms, Leonardo will visit many science centers in Egypt, Greece, Sicily, Syria. In this way he mastered both the mathematics practiced by scientists and the popular mathematics of traders.

Around 1200, after completing his travels, he returned to Italy and wrote a number of books in which he laid down the knowledge gained during his travels in the Mediterranean world.

Among the published books will be The Book of Squares (Liber quadratorum) -

probably the most valuable, but also the least popular, The Practice of Geometry (Practica geometriae), a commentary, now lost, on Euclid's Book X (Book of Irrationals), as well as Liber abacci (The Book of the Abacus), which is also the most popular.

Written in 1202 and revised in 1228, Liber abacci, contrary to the title, is not a book about abacus, but about reckoning, through which Europeans became acquainted with the Arabic numeral system. In the 1228 edition, in a problem, the famous string appears, which today is called the Fibonacci string. This series appears as a solution to a problem related to a hypothetical population of rabbits, in which the chicks do not die, and each pair gives birth to a new pair. Here is the statement of the problem, as given by Fibonacci:

A man put a pair of rabbits in a place surrounded by a wall. How many pairs of rabbits will there be after a year, knowing that every month a pair of rabbits gives birth to a new pair, and a new pair can give birth from the second month.

The first two terms are obviously 1 and 1, and each subsequent term is the sum of the previous two. So it is the string 1, 1, 2, 3, 5, 8, 13, 21,… and so on.

It is found that after 12 months we will have 377 pairs.

Subsequently, countless properties of this string have been discovered, as well as the fact that it is frequently found in nature. In the 19th century, the mathematician Edouard Lucas (1842-1891) gave the name of this string as the Fibonacci string and found many properties of the string. In 1962, the Fibonacci Society was founded, which since 1963 has published a quarterly magazine, The Fibonacci Quarterly, dedicated to revealing the secrets of these issues.

In 1220, Fibonacci published, Practica geometriae, which increased his reputation. This manuscript deals with problems of practical geometry and the measurement of objects. Here he demonstrates a deep knowledge of Euclid's work. He soon attracted the attention of Emperor Frederick II (1194-1250), a famous patron of science, who, visiting Pisa in 1225, refereed a competition among the mathematicians of the time, won detached from the Fibonacci. On this occasion Fibonacci surprisingly solved a third degree equation, finding (it is not known how) the solution with 9 exact decimals: 1022I7II42III33IV4V40VI (in sexagesimal writing) or $1+\frac{22}{60}+\frac{7}{3600}+\frac{42}{216000}+\ldots$ in current writing.

In 1225 Fibonacci will write Liber quadratorum (Book of Squares), a work dedicated to the emperor. In the book he presents a number of problems with indeterminate equations of degrees I and II and resumes the problems he faced in early competitions. Also in this treatise, Fibonacci describes a method of obtaining Pythagorean triplets. In this book, Fibonacci also proves that there are no two squares of integers so that their sum and their difference are perfect squares.

Fibonacci, who enjoyed great recognition during his lifetime and whose fame grew in the centuries that followed, died in 1250, during a war between Pisa and Genoa.

The area of study known as the history of mathematics is primarily an investigation into the origin of discoveries in mathematics and, to a lesser extent, an investigation into the mathematical methods and notation of the past. Before the modern age and the worldwide spread of knowledge, written examples of new mathematical developments have come to light only in a few locales. From 3000 BC the Mesopotamian states of Sumer, Akkad and Assyria, followed closely by Ancient Egypt and the Levantine state of Ebla began using arithmetic, algebra and geometry for purposes of taxation, commerce, trade and also in the patterns in nature, the field of astronomy and to record time and formulate calendars.

The earliest mathematical texts available are from Mesopotamia and Egypt – Plimpton 322 (Babylonian c. 2000 – 1900 BC), the Rhind Mathematical Papyrus (Egyptian c. 1800 BC)and the Moscow Mathematical Papyrus (Egyptian c. 1890 BC). All of these texts mention the so-called Pythagorean triples, so, by inference, the Pythagorean theorem seems to be the most ancient and widespread mathematical development after basic arithmetic and geometry.

The study of mathematics as a "demonstrative discipline" began in the 6th century BC with the Pythagoreans, who coined the term "mathematics" from the ancient Greek μάθημα (mathema), meaning "subject of instruction".Greek mathematics greatly refined the methods (especially through the introduction of deductive reasoning and mathematical rigor in proofs) and expanded the subject matter of mathematics. Although they made virtually no contributions to theoretical mathematics, the ancient Romans used applied mathematics in surveying, structural engineering, mechanical engineering, bookkeeping, creation of lunar and solar calendars, and even arts and crafts. Chinese mathematics made early contributions, including a place value system and the first use of negative numbers.The Hindu–Arabic numeral system and the rules for the use of its operations, in use throughout the world today evolved over the course of the first millennium AD in India and were transmitted to the Western world via Islamic mathematics through the work of Muḥammad ibn Mūsā al-Khwārizmī.

Islamic mathematics, in turn, developed and expanded the mathematics known to these civilizations. Contemporaneous with but independent of these traditions were the mathematics developed by the Maya civilization of Mexico and Central America, where the concept of zero was given a standard symbol in Maya numerals.

Many Greek and Arabic texts on mathematics were translated into Latin from the 12th century onward, leading to further development of mathematics in Medieval Europe. From ancient times through the Middle Ages, periods of mathematical discovery were often followed by centuries of stagnation. Beginning in Renaissance Italy in the 15th century, new mathematical developments, interacting with new scientific discoveries, were made at an increasing pace that continues through the present day. This includes the groundbreaking work of both Isaac Newton and Gottfried Wilhelm Leibniz in the development of infinitesimal calculus during the course of the 17th century. At the end of the 19th century the International Congress of Mathematicians was founded and continues to spearhead advances in the field.

Prehistoric. The origins of mathematical thought lie in the concepts of number, patterns in nature, magnitude, and form. Modern studies of animal cognition have shown that these concepts are not unique to humans. Such concepts would have been part of everyday life in hunter-gatherer societies. The idea of the "number" concept evolving gradually over time is supported by the existence of languages which preserve the distinction between "one", "two", and "many", but not of numbers larger than two.

The Ishango bone, found near the headwaters of the Nile river (northeastern Congo), may be more than 20,000 years old and consists of a series of marks carved in three columns running the length of the bone. Common interpretations are that the Ishango bone shows either a tally of the earliest known demonstration of sequences of prime numbers or a six-month lunar calendar. Peter Rudman argues that the development of the concept of prime numbers could only have come about after the concept of division, which he dates to after 10,000 BC, with prime numbers probably not being understood until about 500 BC. He also writes that "no attempt has been made to explain why a tally of something should exhibit multiples of two, prime numbers between 10 and 20, and some numbers that are almost multiples of 10. The Ishango bone, according to scholar Alexander Marshack, may have influenced the later development of mathematics in Egypt as, like some entries on the Ishango bone, Egyptian arithmetic also made use of multiplication by 2; this however, is disputed.

Predynastic Egyptians of the 5th millennium BC pictorially represented geometric designs. It has been claimed that megalithic monuments in England and Scotland, dating from the 3rd millennium BC, incorporate geometric ideas such as circles, ellipses, and Pythagorean triples in their design. All of the above are disputed however, and the currently oldest undisputed mathematical documents are from Babylonian and dynastic Egyptian sources.

Babylonian mathematics refers to any mathematics of the peoples of Mesopotamia (modern Iraq) from the days of the early Sumerians through the Hellenistic period almost to the dawn of Christianity.The majority of Babylonian mathematical work comes from two widely separated periods: The first few hundred years of the second millennium BC

(Old Babylonian period), and the last few centuries of the first millennium BC (Seleucid period). It is named Babylonian mathematics due to the central role of Babylon as a place of study. Later under the Arab Empire, Mesopotamia, especially Baghdad, once again became an important center of study for Islamic mathematics.

Geometry problem on a clay tablet belonging to a school for scribes. In contrast to the sparsity of sources in Egyptian mathematics, knowledge of Babylonian mathematics is derived from more than 400 clay tablets unearthed since the 1850s. Written in Cuneiform script, tablets were inscribed whilst the clay was moist, and baked hard in an oven or by the heat of the sun. Some of these appear to be graded homework.

The earliest evidence of written mathematics dates back to the ancient Sumerians, who built the earliest civilization in Mesopotamia. They developed a complex system of metrology from 3000 BC. From around 2500 BC onward, the Sumerians wrote multiplication tables on clay tablets and dealt with geometrical exercises and division problems. The earliest traces of the Babylonian numerals also date back to this period.

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Babylonian mathematics were written using a sexagesimal (base-60) numeral system. From this derives the modern-day usage of 60 seconds in a minute, 60 minutes in an hour, and 360 (60 × 6) degrees in a circle, as well as the use of seconds and minutes of arc to denote fractions of a degree. It is likely the sexagesimal system was chosen because 60 can be evenly divided by 2, 3, 4, 5, 6, 10, 12, 15, 20 and 30. Also, unlike the Egyptians, Greeks, and Romans, the Babylonians had a place-value system, where digits written in the left column represented larger values, much as in the decimal system.

The power of the Babylonian notational system lay in that it could be used to represent fractions as easily as whole numbers; thus multiplying two numbers that contained fractions was no different from multiplying integers, similar to modern notation. The notational system of the Babylonians was the best of any civilization until the Renaissance, and its power allowed it to achieve remarkable computational accuracy; for example, the Babylonian tablet YBC 7289 gives an approximation of $\sqrt{2}$ accurate to five decimal places. The Babylonians lacked, however, an equivalent of the decimal point, and so the place value of a symbol often had to be inferred from the context. By the Seleucid period, the Babylonians had developed a zero symbol as a placeholder for empty positions; however it was only used for intermediate positions. This zero sign does not appear in terminal

positions, thus the Babylonians came close but did not develop a true place value system.

Other topics covered by Babylonian mathematics include fractions, algebra, quadratic and cubic equations, and the calculation of regular numbers, and their reciprocal pairs. The tablets also include multiplication tables and methods for solving linear, quadratic equations and cubic equations, a remarkable achievement for the time. Tablets from the Old Babylonian period also contain the earliest known statement of the Pythagorean theorem. However, as with Egyptian mathematics, Babylonian mathematics shows no awareness of the difference between exact and approximate solutions, or the solvability of a problem, and most importantly, no explicit no explicit statement of the need for proofs or logical principles

Egyptian mathematics refers to mathematics written in the Egyptian language. From the Hellenistic period, Greek replaced Egyptian as the written language of Egyptian scholars. Mathematical study in Egypt later continued under the Arab Empire as part of Islamic mathematics, when Arabic became the written language of Egyptian scholars.

The most extensive Egyptian mathematical text is the Rhind papyrus (sometimes also called the Ahmes Papyrus after its author), dated to c. 1650 BC but likely a copy of an older document from the Middle Kingdom of about 2000–1800 BC.It is an instruction manual for students in arithmetic and geometry. In addition to giving area formulas and methods for multiplication, division and working with unit fractions, it also contains evidence of other mathematical knowledge, including composite and prime numbers; arithmetic, geometric and harmonic means; and simplistic understandings of both the Sieve of Eratosthenes and perfect number theory (namely, that of the number 6). It also shows how to solve first order linear equations as well as arithmetic and geometric series.

Another significant Egyptian mathematical text is the Moscow papyrus, also from the Middle Kingdom period, dated to c. 1890 BC. It consists of what are today called *word problems* or *story problems*, which were apparently intended as entertainment. One problem is considered to be of particular importance because it gives a method for finding the volume of a frustum (truncated pyramid).

Finally, the Berlin Papyrus 6619 (c. 1800 BC) shows that ancient Egyptians could solve a second-order algebraic equation.

Greek mathematics refers to the mathematics written in the Greek language from the time of Thales of Miletus (~600 BC) to the closure of the Academy of Athens in 529 AD. Greek mathematicians lived in cities spread over the entire Eastern Mediterranean, from Italy to North Africa, but were united by culture and language. Greek mathematics of the period following Alexander the Great is sometimes called Hellenistic mathematics.

Greek mathematics was much more sophisticated than the mathematics that had been developed by earlier cultures. All surviving records of pre-Greek mathematics show the use of inductive reasoning, that is, repeated observations used to establish rules of thumb. Greek mathematicians, by contrast, used deductive reasoning. The Greeks used logic to derive conclusions from definitions and axioms, and used mathematical rigor to prove them.

Greek mathematics is thought to have begun with Thales of Miletus (c. 624–c.546 BC) and Pythagoras of Samos (c. 582–c. 507 BC). Although the extent of the influence is disputed, they were probably inspired by Egyptian and Babylonian mathematics. According to legend, Pythagoras traveled to Egypt to learn mathematics, geometry, and astronomy from Egyptian priests. Thales used geometry to solve problems such as calculating the height of pyramids and the distance of ships from the shore. He is credited with the first use of deductive reasoning applied to geometry, by deriving four corollaries to Thales' Theorem. As a result, he has been hailed as the first true mathematician and the first known individual to whom a mathematical discovery has been attributed.

Pythagoras established the Pythagorean School, whose doctrine it was that mathematics ruled the universe and whose motto was "All is number". It was the Pythagoreans who coined the term "mathematics", and with whom the study of mathematics for its own sake begins. The Pythagoreans are credited with the first proof of the Pythagorean theorem, though the statement of the theorem has a long history, and with the proof of the existence of irrational numbers. Although he was preceded by the Babylonians and the Chinese, the Neopythagorean mathematician Nicomachus (60–120 AD) provided one of the earliest Greco-Roman multiplication tables, whereas the oldest extant Greek multiplication table is found on a wax tablet dated to the 1st century AD (now found in the British Museum).The association of the Neopythagoreans with the Western invention of the multiplication table is evident in its later Medieval name: the mensa Pythagorica.

Plato (428/427 BC – 348/347 BC) is important in the history of mathematics for inspiring and guiding others. His Platonic Academy, in Athens, became the mathematical center of the world in the 4th century BC, and it was from this school that the leading mathematicians of the day, such as Eudoxus of Cnidus, came. Plato also discussed the foundations of mathematics, clarified some of the definitions (e.g. that of a line as "breadthless length"), and reorganized the assumptions. The analytic method is ascribed to Plato, while a formula for obtaining Pythagorean triples bears his name.

Eudoxus (408–c. 355 BC) developed the method of exhaustion, a precursor of modern integration and a theory of ratios that avoided the problem of incommensurable magnitudes. The former allowed the calculations of areas and volumes of curvilinear figures, while the latter enabled subsequent geometers to make significant advances in geometry. Though he made no specific technical mathematical discoveries, Aristotle (384–c. 322 BC) contributed significantly to the development of mathematics by laying the foundations of logic.

One of the oldest surviving fragments of Euclid's Elements, found at Oxyrhynchus and dated to circa AD 100. The diagram accompanies Book II, Proposition 5.

In the 3rd century BC, the premier center of mathematical education and research was the Musaeum of Alexandria. It was there that Euclid (c. 300 BC) taught, and wrote the Elements, widely considered the most successful and influential textbook of all time.The Elements introduced mathematical rigor through the axiomatic method and is the earliest example of the format still used in mathematics today, that of definition, axiom, theorem, and proof. Although most of the contents of the Elements were already known, Euclid arranged them into a single, coherent logical framework.

The Elements was known to all educated people in the West up through the middle of the 20th century and its contents are still taught in geometry classes today. In addition to the familiar theorems of Euclidean geometry, the Elements was meant as an introductory textbook to all mathematical subjects of the time, such as number theory, algebra and solid geometry, including proofs that the square root of two is irrational and that there are infinitely many prime numbers. Euclid also wrote extensively on other subjects, such as conic sections, optics, spherical geometry, and mechanics, but only half of his writings survive. Archimedes (c. 287–212 BC) of Syracuse, widely considered the greatest mathematician of antiquity,[58] used the method of exhaustion to calculate the area under the arc of a parabola with the summation of an infinite series, in a manner not too dissimilar from modern calculus. He also showed one could use the method of exhaustion to calculate the value of π with as much precision as desired, and obtained the most accurate value of π then known. He also studied the spiral bearing his name, obtained formulas for the volumes of surfaces of revolution (paraboloid, ellipsoid, hyperboloid), and an ingenious method of exponentiation for expressing very large numbers. While he is also known for his contributions to physics and several advanced mechanical devices, Archimedes himself placed far greater value on the products of his thought and general mathematical principles. He regarded as his greatest achievement his finding of the surface area and volume of a sphere, which he obtained by proving these are 2/3 the surface area and volume of a cylinder circumscribing the sphere.

Apollonius of Perga (c. 262–190 BC) made significant advances to the study of conic sections, showing that one can obtain all three varieties of conic section by varying the angle of the plane that cuts a double-napped cone. He also coined the terminology in use today for conic sections, namely parabola ("place beside" or "comparison"), "ellipse" ("deficiency"), and "hyperbola" ("a throw beyond"). His work Conics is one of the best known and preserved mathematical works from antiquity, and in it he derives many theorems concerning conic sections that would prove invaluable to later mathematicians and astronomers studying planetary motion, such as Isaac Newton. While neither Apollonius nor any other Greek mathematicians made the leap to coordinate geometry, Apollonius' treatment of curves is in some ways similar to the modern treatment, and some of his work seems to anticipate the development of analytical geometry by Descartes some 1800 years later.

Around the same time, Eratosthenes of Cyrene (c. 276–194 BC) devised the Sieve of Eratosthenes for finding prime numbers.[68] The 3rd century BC is generally regarded as the "Golden Age" of Greek mathematics, with advances in pure mathematics henceforth in relative decline. Nevertheless, in the centuries that followed significant advances were made in applied mathematics, most notably trigonometry, largely to address the needs of astronomers.[69] Hipparchus of Nicaea (c. 190–120 BC) is considered the founder of trigonometry for compiling the first known trigonometric table, and to him is also due the systematic use of the 360 degree circle. Heron of Alexandria (c. 10–70 AD) is credited with Heron's formula for finding the area of a scalene triangle and with being the first to recognize the possibility of negative numbers possessing square roots. Menelaus of Alexandria (c. 100 AD) pioneered spherical trigonometry through Menelaus' theorem. The most complete and influential trigonometric work of antiquity is the Almagest of Ptolemy (c. AD 90–168), a landmark astronomical treatise whose trigonometric tables would be used by astronomers for the next thousand years. Ptolemy is also credited with Ptolemy's theorem for deriving trigonometric quantities, and the most accurate value of π outside of China until the medieval period, 3.1416.

Project funded by: Erasmus+ / Key Action 2 - Cooperation for innovation and the exchange of good practices, Strategic Partnerships for school education (European Commission, EACEA)

Title page of the 1621 edition of Diophantus' Arithmetica, translated into Latin by Claude Gaspard Bachet de Méziriac.

Following a period of stagnation after Ptolemy, the period between 250 and 350 AD is sometimes referred to as the "Silver Age" of Greek mathematics. During this period, Diophantus made significant advances in algebra, particularly indeterminate analysis, which is also known as "Diophantine analysis". The study of Diophantine equations and Diophantine approximations is a significant area of research to this day. His main work was the Arithmetica, a collection of 150 algebraic problems dealing with exact solutions to determinate and indeterminate equations. The Arithmetica had a significant influence on later mathematicians, such as Pierre de Fermat, who arrived at his famous Last Theorem after trying to generalize a problem he had read in the Arithmetica (that of dividing a square into two squares). Diophantus also made significant advances in notation, the Arithmetica being the first instance of algebraic symbolism and syncopation.

The Hagia Sophia was designed by mathematicians Anthemius of Tralles and Isidore of Miletus.

Among the last great Greek mathematicians is Pappus of Alexandria (4th century AD). He is known for his hexagon theorem and centroid theorem, as well as the Pappus configuration and Pappus graph. His Collection is a major source of knowledge on Greek mathematics as most of it has survived. Pappus is considered the last major innovator in Greek mathematics, with subsequent work consisting mostly of commentaries on earlier work.

The first woman mathematician recorded by history was Hypatia of Alexandria (AD 350–415). She succeeded her father (Theon of Alexandria) as Librarian at the Great Library[citation needed] and wrote many works on applied mathematics. Because of a political dispute, the Christian community in Alexandria had her stripped publicly and executed. Her death is sometimes taken as the end of the era of the Alexandrian Greek mathematics, although work did continue in Athens for another century with figures such as Proclus, Simplicius and Eutocius. Although Proclus and Simplicius were more philosophers than mathematicians, their commentaries on earlier works are valuable sources on Greek mathematics. The closure of the neo-Platonic Academy of Athens by the emperor Justinian in 529 AD is traditionally held as marking the end of the era of Greek mathematics, although the Greek tradition continued unbroken in the Byzantine empire with mathematicians such as Anthemius of Tralles and Isidore of Miletus, the architects of the Hagia Sophia. Nevertheless, Byzantine mathematics consisted mostly of commentaries, with little in the way of innovation, and the centers of mathematical innovation were to be found elsewhere by this time.

The history of mathematics and mathematics education in Romania in the XVII - XIX centuries In Romania, mathematics appeared and developed with the advent of organized education, school. A beginning of mathematical education is signaled in the 16th century, by the establishment of the first School from Cotnari, where mathematics was taught. This school had a gymnasium degree, not an academy which is synonymous with university. In the 17th century, the first Greek academies appeared in Iaşi and Bucharest, founded by Vasile Lupu and Constantin Brâncoveanu, respectively. Both academies lasted until 1821 when the Turks realized that they were propagandizing against their empire. In these academies, in the last three years of study (10, 11, 12) they are taught in Greek: practical and rational arithmetic, algebra, the theory and practice of algorithms, plane and spherical trigonometry, astronomy and the application of mathematics to military art. It should be noted that in addition to these academies, both in Muntenia and in Moldova there were other schools (in which elementary knowledge is taught: reading, writing and counting), generally in addition to monasteries and churches. An important development of mathematics education took place at the beginning of the 19th century with the founding in August 1818 of the Engineering School of Gheorghe Lazăr (1779 - 1823) from Sf. Sava (Bucharest) and in Iaşi of the School of Engineering. founded by Gheorghe Asachi (1788 - 1869) with teaching in French at the beginning. Both Lazăr and Asachi wrote books on: arithmetic, geometry, trigonometry, in Romanian, after which they studied for many years in schools in the first half of the 19th century. It has been convinced that it is possible to learn in the mother tongue, no matter how high the studies to be done, a fact with positive consequences on education in general and mathematics in particular, and later on mathematical creation.

The first original Romanian contribution in mathematics was that of Dimitrie Asachi (1820 - 1868), the son of Gheorghe Asachi, officer and engineer. This work was published in Munich in 1841 and refers to the inversion of the series. After the Union of the Romanian Principalities in 1859, Alexandru Ioan Cuza established universities in Iaşi in 1860 and in Bucharest in 1864, with several faculties, including the Faculty of Physical, Mathematical and Natural Sciences. Among the mathematics professors from the newly established universities who produced and published original works we mention Emanoil Bacaloglu (1830 - 1891) from the University of Bucharest who introduced a curve that bears his name in the theory of surfaces, and Neculai Şt. Baptism (1843 - 1920) at the University of Iasi with a paper on the harmonic series.

All mathematics professors at both universities have made important contributions to the establishment of mathematics education on a solid footing and to the training of specialists in railways, bridges, buildings, the chemical industry, and so on. Most of these teachers have published textbooks for both secondary and higher education. Thus, the first work for students in mathematics was "Differential and integral calculus" by Neculai Culianu (Iasi, 1870), followed by "Course of analytical geometry" by C. Climescu (Iasi, 1898). University professors have published mathematics journals to raise the level of secondary and higher mathematics education.

Thus, "Scientific Recreations" appeared in 1883 in Iasi and "Mathematical Gazette" in 1895 in Bucharest. From

the pleiad of illustrious mathematics professors from the University of Bucharest we will focus on two, who influenced for a long time the education in our country and who also brought original contributions in mathematics: Spiru C. Haret (1851 - 1912). He was born on February 15, 1851 in Hanul Conachi in Dorohoi. He attended primary school in his parents' house, then at a school in Dorohoi and in September 1862 he entered the St. Sava high school as a scholarship holder (he was a poor child, 45 but very gifted for study). In 1869, after graduating from high school, he enrolled at the University of Bucharest at the Faculty of Sciences, physics and mathematics. In December 1870, although a second year student, he obtained the chair of mathematics at the Central Seminary by competition. After a year he gave up the chair and continued his studies at the University. He graduated in 1874, at the age of 23.

Titu Maiorescu, the Minister of Public Instruction, grants him a scholarship, based on a competition, to study mathematics in Paris. Here, realizing his shortcomings in mathematics, especially his problem-solving skills, he again spent his bachelor's degree in mathematics and then in 1878 defended his doctoral dissertation "On the invariability of the major axes of planetary orbits." carrying on and correcting the research of Laplace, Lagrange and Poisson on the variety of axes of planetary orbits. Haret emphasizes the "pure secular terms" and for the "third degree", which presented in a different light the stability of the planetary system.

The scholar mathematician and astronomer Jules Henri Poincaré observed: "In 1878 Spiru Haret proved the existence of secular terms of the third degree and this result caused great astonishment." In 1885, Haret's doctoral dissertation was republished in the Annals of the Astronomical Observatory in Paris. The Faculty of Sciences in Paris sends an address to the Ministry of Cults and Public Instruction congratulating Romania, the country that produced and possesses such talents. Later, in 1976, on the occasion of the 125th anniversary of Spiru Haret's birth, a crater on the map of the Moon, on the coordinates: latitude 59 degrees south and longitude 176 degrees west, in the invisible lunar part, was named

Haret. He was the first Romanian to announce the value, later confirmed, of the Romanian school of mathematics, becoming the first Romanian doctor of mathematics in Paris. Spiru Haret could remain a university professor in France.

However, he preferred the Faculty of Sciences in Bucharest, where he became a professor since 1878 following a brilliant competition (since 1882 professor of analytical geometry at the School of Bridges and Roads). He taught until 1910, when he retired, and even after that, until his death, giving popularization lectures at the People's University. In 1910 he published "Social Mechanics, in Paris and Bucharest, using for the first time, mathematics in explaining and understanding social phenomena. Haret had a prodigious activity for raising the level of Romanian education on all its levels: primary, secondary and university. He was the undisputed soul of the Romanian school between 1880 and 1910 and that is why he was called "the man of the school". The guiding principle in terms of education was the "primacy of the school" and it managed to raise the level of the Romanian school, especially in terms of mathematics programs, to the level of the most advanced countries.

As a teacher, Spiru Haret had the gift of exposition, with simple, intuitive demonstrations, illustrated with various practical applications. His explanations were clear, flowing, systematic and logical. As a man he was modest, silent, honest, upright, and determined. He was one of the few who helped inventor Aurel Vlaicu. He died on December 17, 1912, of cancer, after as a member of the Romanian Academy, in the meeting of May 18, 1912, he presented the communication "The big red spot on the planet Jupiter". Spiru Haret remained in science through two original works: his doctoral thesis and "Social Mechanics", published in 1910 in French and translated into Romanian in 1969, but in the history of culture and education is considered the founder of the modern Romanian school. It can be said that Spiru Haret as a mathematician was concerned with the "problem of planetary stability", and as a man of the school, he was concerned with its progress.

David Emmanuel (1854 - 1941). He was born on January 31, 1854 in Bucharest, to very poor parents, his father, Manole Emmanuel, being a carpenter. He attended primary school in Ploieşti, between 1861 and 1864, where his parents lived. He took the first four secondary classes at the Şincai Gymnasium in Bucharest, between 1865 and 1869; then he went to the high school Gh. Lazăr, where he studied the whole course between 1869 and 1873. With the money he raised from meditations, in the higher course he went to Paris to study at the Sorbonne without a scholarship. In Paris he graduated in mathematics and later in physics. He is defending his doctoral thesis in mathematics "Etude des intégrales abéliennes des troisième espèce". He is the second Romanian citizen to have a doctorate in mathematics from Paris. 47 Returning to the country in the autumn of 1879 he was employed as a professor of mathematics at a high school, and in 1880 a professor at the Faculty of Sciences of the University of Bucharest and the School of Bridges and Roads, and from 1882 professor at the Normal High School where he taught for for the first time in our country group theory and Galois theory. In the university course, in the general theory of analytical functions, David Emmanuel was a follower of Weierstrass, with a tendency to arithmetic in analysis.

The course introduces the notion of analytical functions through Weierstrass's series of powers. In the third year course, Emmanuel treated elliptic functions as a result of the inversion of elliptic integrals, and in the final part of the course he dealt with the latest developments of that time (for example Emile Picard's two famous theorems on integer analytic functions). The courses that David Emmanuel took - "old David" or "father David" as the students called him - at the university or at the polytechnic were methodical, precise, clear, full of order and richness of facts, full of harmony and constantly up to date. news on the subject. Because he followed the continuous evolution of the disciplines he taught, the courses, especially the theory of functions, were always renewed.

In 1907, together with Spiru Haret, D. Emmanuel was celebrated at the university and at the School of Bridges and Roads to celebrate 25 years of university teaching. On

September 15, 1929, it was celebrated for 50 years since the defense of the thesis and 48 years of university teaching, by all Romanian mathematicians. On May 25, 1936, he was elected an honorary member of the Romanian Academy. Serene and modest like the ancient philosophers (he had a splendid classical culture, which was often betrayed in his most difficult mathematical expositions), clear and precise in his exposition, he always highlighted the essence of a problem posed or of a rigorous demonstration. He was a prestigious man like S. Haret, esteemed for his science and character. He died in Bucharest, at the age of 87, on February 4, 1941.

D. Emmanuel will remain in the history of mathematics as one of those who, practicing mathematics at a high level for 48 years, made premises for the emergence of the Romanian mathematics school. All our leading mathematicians of May 48, such as łițeica, Pompei, Lalescu, etc., acknowledged that if you have reached important creations in mathematics, it is due to the granite foundation laid by this education by the great pedagogue David Emmanuel.

D. Emmanuel's mathematical work is quantitatively reduced, but qualitatively it is of some value. Among the didactic works are: "Course of infinitesimal analysis" (1925) and "Lessons of the theory of functions", in two parts published by Casa şcoalelor Publishing House, 1924 and 1927. As characterized by Pangrati at a celebration, David Emmanuel, charming teacher , far from vain turmoil and regardless of glory, remains "a sage."

History of mathematics and mathematics education in Romania in the twentieth century At the beginning of the twentieth century, following the Haret law reforming education in 1898, with numerous doctors in mathematics (from the west): Haret, Emmanuel, Țițeica, Constantin Gogu, Nicolae Coculescu, Anton Davidoglu, D. Pompeiu, Traian Lalescu and others. and with a prestigious magazine "Gazeta matematică", it was normal for us to end up creating in mathematics, and after 1920 to be able to speak even of a "Romanian mathematics school" recognized abroad. At one point, six memoirs written by Romanian mathematicians appeared in a single issue of Comptes rendus de l'Académie française des sciences.

The emergence of schools and universities in the second half of the nineteenth century boosted our mathematical education and even mathematical creation, but the gap with many European countries being enormous.

Universities with a past of over 800 years (Bologna created in 1088) from abroad influenced the development of education in those countries, while in our country only in the second half of the nineteenth century appear, and in the third quarter of the nineteenth century - we begin to be recorded with original works of mathematics (the works of Em. Bacaloglu and N. Şt. Botez).

In the fourth quarter of the nineteenth century appear the first 49 doctors of mathematics at the Sorbonne and the two publications: Scientific Recreations in Iasi and the Mathematical Gazette in Bucharest, all creating the premises for a boost of mathematics education, able to reach followed that of the advanced countries.

Finally, with the beginning of the 20th century, an epoch of mathematical creation proper is outlined, in which a Romanian mathematical school known as such abroad is formed. In such a short time, creators like łițeica, Pompeiu and Lalescu appeared. They were also known abroad, łițeica and Pompeiu giving classes at the Sorbonne, being cited in many specialized treatises, and in certain mathematical fields they were pioneers. Around them were formed our mathematicians from the first half of the twentieth century and who continued their creative work in the second half of the century.

For example, geometers were formed around Țițeica: Gh. Vrănceanu, Dan Barbilian, N. Mihăileanu and others, at Pompeiu's analysis seminar: Simion Stoilow, Miron Nicolescu, Alexandru Froda, Octav Onicescu and others ., and around Lalescu, who unfortunately had a short life, swarmed a series of mathematicians with concerns about algebra and the theory of integral equations.

Below we record some data about the life and work of these three parents of Romanian mathematics. Gheorghe Țițeica (1873 - 1939) was born on October 14/16, 1873 in Turnu Severin. After graduating in 1895 from the mathematics department, the Faculty of Sciences, University of Bucharest, went to Paris in 1896 to study mathematics at the École Normale Supérieure (Sorbonne), where in 1899 he defended his doctoral dissertation before a committee chaired by Gaston Darboux.

Țițeica's scientific work includes 96 scientific memoirs, most of them projective and affine differential geometry. Dimitrie Pompeiu (1873 - 1954) was born on September 22 / October 4, 1873 in the village of Dimăcheni (from Dorohoi), as the son of the teacher Dimitrie Pompeiu, a former classmate at the high school in Botoşani with Eminescu. Between 1889 and 1893 he attended the School of Teachers in Bucharest. In 1898 he left for Paris to complete 50 studies in mathematics. After a year of study, he passed his French baccalaureate, and in 1899 he enrolled at the Sorbonne, where, in 1903, he obtained a degree in mathematics. In 1905 he defended his doctoral dissertation before a commission chaired by Poincaré.

Pompey's scientific activity took place in four areas: function theory and functional calculus, set theory, rational mechanics and differential and integral calculus. His mathematical work includes 129 memoirs in prestigious mathematical periodicals in France, Belgium, Germany, Italy, the United States, Japan, Portugal, the Netherlands, etc. He had a special predilection for the theory of functions, introducing several new notions of great value, among which we mention: Pompeii functions, areolar derivative, theorem of finite increases in the complex field, various functional equations, the fundamentals of mechanics. But Pompey often focused on the problems of elementary mathematics, especially geometry.

A typical case is the famous theorem: "the distances from a point in the plane of an equilateral triangle to its sides are the sides of a triangle", a theorem that bears his name. Traian Lalescu (1882 - 1929) was born on July 24, 1882 in Bucharest. In the sixth grade of high school he became a correspondent for the Mathematical Gazette with an extremely prodigious activity. After graduating from high school in 1900, he first entered the School of Bridges and Roads in Bucharest, following its courses until 1903 when he retired and moved permanently to the Faculty of Sciences, mathematics department of the University of Bucharest. In June 1903 he graduated in mathematics and obtained a scholarship to the Sorbonne. In 1908 he defended his doctoral thesis in mathematics under the presidency of E. Picard.

The results of the doctoral thesis impressed so many that they were cited in the great courses of mathematical analysis. At the World Congress of Mathematics in Rome from March 23 to 30, 1908, he met the famous mathematician Vito Voltera (1860 - 1940) with whom he established a true scientific collaboration. In Lalescu's mathematical creation, there are concerns about number theory and algebra, geometry, vector and tensor calculus, mathematical analysis, mechanics, electricity.

He has published over 100 articles and memoirs in various prestigious magazines in the country and abroad, especially France. Traian Lalescu was the Romanian mathematician of a rare strength, conception, spontaneity and originality. He was the mathematician of generalizations, of elegant and simple solutions.

These three great founders of the Romanian mathematics school formed through their lessons, through their scientific research works as well as through their dedication to support the development of mathematics education in our country a series of mathematics that turned into a real Romanian mathematical phenomenon.

A. Analysis and theory of functions

1. Anton Davidoglu

(1873 - 1958), born in Bârlad, did his higher studies and doctorate at the Sorbonne in 1900. He published 12 memoirs and two didactic works.

2. Theodor Angheluță (1882 - 1964), born in a village in the former Tutova county, studied in Bucharest and the Sorbonne, and his doctorate in Bucharest in 1922. He has published over 70 scientific papers, as well as a series of university courses of great clarity.

3. Aurel Angelescu (1886 - 1938), born in Ploieşti, did his higher studies and doctorate at the Sorbonne in 1916. He published over 50 scientific papers.

4. Simion Stoilow (1887 - 1961), born in Bucharest, did his higher studies and doctorate at the Sorbonne in 1916. His mathematical work was published in French by the Romanian Academy Publishing House in 1964.

5. Florin Vasilescu (1897 - 1958), born in Călăraşi, did his doctorate at the Sorbonne in 1925. He published 45 memoirs and monographs in various magazines in the country and especially in France.

6. Mihail Ghermănescu (1899 - 1962), born in Bucharest, did his university studies in Bucharest and his doctorate in Cluj in 1933.

7. Alexandru Ghika (1902 - 1964), born in Bucharest, studied in Bucharest and Paris and his doctorate at the Sorbonne in 1927. His scientific activity concerns in particular the functional analysis in which he published over 100 memoirs and didactic works.

8. Miron Nicolescu (1903 - 1975), born in Giurgiu, did his university studies in Bucharest and his doctorate at the Sorbonne in 1928. The scientific activity 52 regards is immense and includes over 150 memoirs published in the most prestigious mathematics

journals in the country and abroad.

9. Adolf Haimovici (1912 -?), Born in Iaşi, studied primary, high school and university and doctorate in Iaşi in 1934. He published over 80 memoirs and a series of didactic works useful to students today.

B. Geometry and topology

1. Alexandru Myller (1879 - 1965), born in Bucharest, studied primary, high school and university in Bucharest, where he graduated in 1900. In 1902 he left for doctoral studies in Germany, first at Berlin and then to Göttingen, where he defended his doctoral dissertation in 1906 before a committee that also included David Hilbert. His mathematical work includes over 120 articles and memoirs published in the country and abroad.

2. Octav Mayer (1895 - 1966), born in Mizil, studied high school and university in Iasi, where he graduated in 1919, after a three-year hiatus due to the war. In 1920 he defended his doctoral thesis in Iaşi, being the first doctor in mathematics at the University of Iaşi and the first doctor in pure mathematics taken in the country. His mathematical work includes over 50 papers published in the country and abroad.

3. Gheorghe Vrănceanu (1900 - 1979), born in Doagele commune, Vaslui county, studied primary in his native village, high school in Vaslui and university in Iaşi, where he graduated in 1922. In 1924 he defended his doctoral thesis in Rome. The scientific work is huge, publishing over 200 memoirs in the most prestigious mathematics publications in the world.

4. Gheorghe Gheorghiev (1907 -), born in the city of Cetatea Albă (Bessarabia), studied at Iaşi. In 1946 he defended his doctoral thesis in Iasi. His mathematical work includes over 70 memoirs of differential geometry and a series of didactic works, among which are "Course of analytical geometry", "Differential geometry" etc.

5. Nicolae N. Mihăileanu (1912 - 1997), born in Constanța , attends primary and high school courses in Constanța, and university courses in Bucharest. In 1949, he defended his doctoral thesis in Bucharest in front of a commission chaired by Gh. Vrănceanu. He has published over 120 memoirs and teaching papers.

C. Algebra and number theory

1. Dan Barbilian (1895 - 1961), born in Câmpulung - Muscel. In 1929 he defended his doctoral thesis in Bucharest. He has published over 120 papers in the most prestigious mathematical publications in the country and abroad, as well as some monographs (courses) that are still relevant today.

2. Ion Creangă (1911 -?), Born in Adâncata, former Dorohoi County. In 1913 he obtained a degree in mathematics in Iasi and in 1939 he received his doctorate in Rome. He has published over 50 papers and has written several didactic papers consulted by students today.

3. Alexandru Froda (1894 - 1973), born in Bucharest, studied primary and high school in Bucharest, in 1912 he entered the National School of Bridges and Roads, which he graduated in 1918, after 3 years of interruption due to the war. He then enrolled at the Faculty of Sciences, Department of Mathematics, of the University of Bucharest, which he graduated in 1927. In 1929 he left for Paris, where, at the end of the same year, he received his doctorate in mathematics at the Sorbonne. He has published over 60 scientific and didactic papers.

D. Pure and applied mathematics

1. Victor Vâlcovici (1885 - 1970), born in Galați, attended primary school and high school in Brăila, then the Faculty of Sciences at the University of Bucharest, which he graduated in 1907. In 1909 he obtained a scholarship for doctorate at Göttingen, and in

1913 he obtained his doctorate with a subject in mechanics. He published 169 memoirs.

2. Caius Iacob (1912 - 1992), born in Arad, attended primary school and the lower course of high school in Arad, and the upper course of high school in Oradea. He attended the Faculty of Sciences at the University of Bucharest, graduating in 1931. In 1935 he obtained his doctorate with a subject in fluid mechanics at the Sorbonne. He has published over 120 scientific papers. 54

3. Mendel Haimovici (1908 - 1973), born in Iaşi and studied in Iaşi. In 1932 he left for Rome and returned in 1933 with a doctorate. He has published over 70 papers in prestigious journals.

4. Octav Onicescu (1892 - 1983), born in Botoşani, attended primary school and high school in Botoşani, and in 1915, in Bucharest, he graduated in parallel from the Faculty of Sciences and the Faculty of Philosophy. He has published over 200 scientific papers, synthesis papers and teaching papers.

5. Gheorghe Mihoc (1906 - 1981), born in Brăila, studied in Bucharest. He graduated in mathematics in 1928, after which he left for Rome, obtaining in 1930 the title of doctor in statistical and actuarial sciences. In 1934 he defended a doctoral thesis in the country. He has published over 100 scientific papers and treatises.

6. Grigore Moisil (1906 - 1973), born in Tulcea, studied in Bucharest, and in parallel with the Faculty of Sciences, he studied between 1924 - 1929 and Politehnica, construction department. In 1929 he defended his doctoral thesis in Bucharest, and in 1931 he taught. He has published over 250 papers, of which 198 are memoirs.

1.1 Definitions and approaches regarding mathematics education at European level

We are living through the educational revolution of the 21st century driven by the advent of new technologies that are undermining the role of the school as the main and exclusive seat of learning. These days, ICTs are now seen as an enhancement to learning, and this conception contributes to their rapid spread and adoption in schools and society. But although the entry of computers into the classroom is now commonplace, this does not always translate into a change in methodology and lesson plans, and thus in the actual teaching curriculum.

So, it is time to start rethinking education by separating it from schooling. Education is a lifelong process that starts at birth and lasts a lifetime, whereas schooling normally takes place between the ages of five and twenty-five.

The lifelong learning is the way to face this age of change and to overcome the barriers still existing between formal, non-formal, and informal education to promote the individual's fulfillment both at an individual and social level.

The determining impulse of the new information technologies is re-evaluating the role that non-formal and informal learning can play within the personal training process and indeed making them become key tools for a school of the future.

ICT is the acronym for Information and Communication Technology, which comes from the union of two terms: information technology and communication technology.

The term ICT is used in a more general sense to indicate the area of activities related to the tools that communicate and disseminate information and news. The use of ICT is enabling a quantum leap in education by creating open and flexible learning environments, capable of breaking down space and time barriers and facilitating the dissemination of knowledge. The student, who was previously seen as a container to be filled with knowledge by passively living the educational process, in these new learning contexts is placed at the center and becomes the protagonist and builder of knowledge himself. They can contribute to the creation of resources and educational content, collaborate and share them with their peers, activating peer-learning.

The educational aims of ICT in schools can be summarised as follows:

- to support computer literacy by guiding the student towards a conscious use of technology;
- to facilitate the teaching-learning process (support for traditional curricular teaching);

- providing new tools
  to support the teacher's professional activity (e.g. introducing new organizational and communication methods inside and outside the school);
- promote collaborative work and study;
- constitute one of the environments for the citizen's cultural development.

The change in educational approach in line with the principles of constructivism has also led to changes in the first teaching practices that made use of technological platforms (Learning Management System) often very rigid that support a type of learning guided by the teacher.

New learning techniques are built around the tools of Web 2.0 tools and exploit its potential; wikis, blogs, social networks, and tools that allow users to interact via the Internet are the infrastructural framework capable of activating pedagogical models based on collaboration and cooperation. The integration of serious games and gamification techniques supported by ICT allow students to engage while keeping their motivation and interest high.

However, it is clear that ICTs are only tools and it is still up to the teacher, whose role remains fundamental as a facilitator of knowledge and skills acquisition. Arzarello's study, for example, highlights the need to focus attention on the role of the teacher. In a didactic design that exploits ICT the teacher has to conduct an analysis of what is to be taught from different points of view: cultural,cognitive, and didactic in order to activate an appropriate 'technological transposition'.

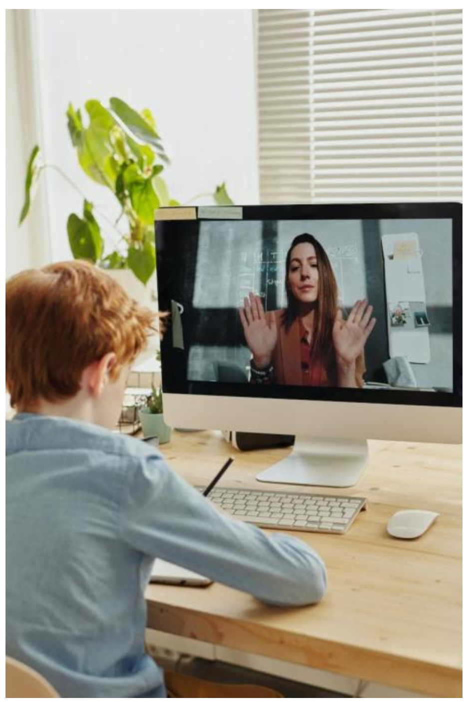
ICT consists of a broad category of hardware and software tools supporting a wide variety of activities. In education some of the technologies that comprise ICT are:

- Personal Computers (notebooks and laptops)
- Smart Devices (Smartphones, tablets)
- Educational software
- Multimedia interactive whiteboards

- Internet
- Simulators
- Learning Platforms

The task of the teacher is to choose the most appropriate technologies to be used in the specific teaching activity to support both the teaching and learning of mathematics.

The new digital teachers must have in-depth knowledge of the new tools available and especially of the methodological and pedagogical approaches capable of exploiting these tools. But above all, as Drijvers studies have shown, the teacher needs a new way of thinking and acting to be able to unhinge the "stability" of traditional teaching practices that are the core of his or her daily activities. This is the real problem that the new generation of teachers, trained no longer in the integration of ICT in daily activities but in their natural and now indispensable daily use, will certainly solve.



Student and digital technologies

"The intrusion of technology into human lives has hit us so fast and with such force, that there has been little time to plan how to accommodate the profound change technology has made to our lives" (Rowan, 2010).

Over the past 20 years, technology has fundamentally changed every aspect of our society, new digital technologies have profoundly changed the way we live our days. All generations, including the older ones, have changed their behavior by adapting to the advantages and limitations that the new ICT tools bring. However, it is the youngest who have been most affected by this digital revolution. Starting with the Millenials (generation of all those born between 1981-1996), individuals have integrated digital devices into their way of life so deeply that it is almost unthinkable to be able to do without them. This process is naturally boosted by the demands of modern society.

Nowadays, there is no educational, social, or work environment in which a good knowledge of new technologies does not lead to a competitive advantage. The new generations are living in symbiosis with their digital devices and integrated with their internet connection have the possibility of finding everything they are interested in with a simple click.

They have thousands of followers on their social networks, create and edit photos and videos with ease and spend most of their time on instant messaging platforms such as Discord and Whatsapp.

Despite the massive presence of technology in their lives, however, many of today's teenagers do not struggle to do a Google search, are unable to create an email account, and have little manual dexterity with keyboard and mouse and with normal productivity tools (Word, Excel, Powerpoint). Many digital natives, who grew up with a smartphone in their hands from a very early age, are incapable of using a personal computer for professional use.

The school's role is therefore to educate young people to make informed use of technology, asserting its role not only as a mere tool for entertainment but as a valid instrument for cultural and professional growth.

1.2 Traditional concepts and methods vs. digitization in schools

a) SEI - Computerized Educational System, is a complex project, initiated by the Ministry of National Education of Romania in 2001, to support the teaching-learning process in pre-university education with state-of-the-art technologies.

In 2001, in Romania there were only 3 computers for every 100 students in middle school and 5 computers for every 100 high school students. On average, for every 100 students, 3.5 computers connected to the Internet returned.
Bringing the level of computerization of education to European standards required with priority the extension of access to computers for each student and for each teacher. In order for such an approach to be successful, it was necessary to endow, at national level,

all the schools in pre-university education.

The Ministry of National Education of Romania has chosen for the implementation of the SEI program a consortium of companies of which SIVECO Romania is part - responsible for the development of educational software, multimedia educational content and software installation and configuration, training and technical support, Microsoft (for applications), IBM and HP for hardware.

The project took place during 6 stages, in the period 2001 - 2009, at national level, following its implementation, each school in Romania benefiting from the latest technologies:

-over 7 million users and beneficiaries involved in the project: students, teachers, administrative staff, parents, decision makers, education specialists

-over 15,000 computerized laboratories equipped with state-of-the-art technology

-140,000 trained teachers

-192,000 desktops and laptops delivered to schools

-3,700 learning lessons, totaling over 21,000 individual learning objects, for 21 subjects connecting to the Internet of over 2,000 schools across the country administration of the SEI educational portal, http://portal.edu.ro, the most important source of information in the educational system, with 228,323 registered users and which registered in July 2013 over 4.23 unique visitors, over 8,500 articles and 2,212 .138 forum posts.

At the same time, within the project, IT projects were developed in order to support the organization of national exams for students in schools and high schools, as well as for teachers:

entrance exams in high schools and vocational schools (ADLIC - approximately 2,489,194 graduates in the period 2001 - 2013); over 2,637,766 candidates for the Baccalaureate (2003-2013); teacher tenure exam (565,378 teachers, 2003-2013).

The SEI project is based on the teaching / learning and content management system, AeL, developed by SIVECO Romania, being intended for use by all categories of participants in the educational process: teachers, pupils, students, management and administrative staff.

The AeL system provides support for teaching, learning, content management and monitoring of results, as well as training and assessment processes, design and monitoring of the use of multimedia content. At the same time, AeL provides the necessary means for communication and synchronization between local and regional centers within the SEI program.

General characteristics

- friendly, flexible interface, differentiated by roles, groups, access rights
- roles, groups, users, and their associated access privileges are very easy to manage
- AeL is compatible with the most recognized standards: MathML, SCORM and IMS
- easy to install and manage
- multi-linguistic and regional support.

b) We live in the century of digital culture, and one of the concerns of the Romanian education system is the continuous adaptation to new trends in terms of modern teaching methods. The integration of modern digital systems, such as the computer or the interactive whiteboard, has become a necessity in the educational process and is no longer seen as a modern phenomenon.

The teacher's role is no longer just to pass on information to students. The challenge now is to capture their attention and maintain their focus throughout the course. At the same time, the teacher must be informed about new technologies in education and modern teaching-learning methods. This is the only way he will be able to understand the younger generations and provide them with quality and attractive information in class. The new generations come into contact from an early age with modern technology, which attracts them like a magnet. The fact that they are accustomed to such small stimuli makes modern digital systems the only solution attractive enough to motivate them, make them active in class and help them retain information more easily.

A product more and more often used in Romanian schools is the interactive whiteboard. It is an easy-to-use tool, based on electromagnetic technology, that increases student involvement and turns class hours into real interactive experiences.

c) Educational applications, tools and platforms

Kinderpedia - Complete communication and management solution for schools and kindergartens, available on the web and as a mobile application (Android and iOS). The platform is free and helps to manage the class / group and learning activities, but also for communication and collaboration. It is adapted for pre-university education and is in Romanian. Allows enrollment of both students and parents.

It has the video-conference mode (Zoom) integrated with the class schedule, but also a catalog of notes with the possibility of assigning and grading homework to students. Students can only register with a unique code provided by the teacher (they do not need email addresses).

Tutorials in Romanian: https://docs.kinderpedia.co/ro/ (21.09.2020)

DigitalEdu - open educational resources - A database with educational resources in digital format created by teachers, organized by disciplines and years of study. Contains links to tests, worksheets, teaching games, movies and simulations. (01.12.2020)

eTwinning - European eTwinning platform, supported by the European Commission. Through eTwinning, teachers can design and carry out educational projects in partnership between several classes in European schools. They can be projects on study or transdisciplinary disciplines. The registration of a project is done from the eTwinning Desktop, by accessing the platform from the personal account (username and password).

Ideas and resources for projects: etwinning.ro/idei-si-resurse-pentru-proiecte/.

A short presentation of eTwinning: eTw_projectare_proiecte.pptx. (05.05.2020)

WAND.education - Allows you to create lessons from scratch (it has countless lesson templates and templates) or take lessons created by other teachers. Easy to use in creating interactive educational content. The lesson and / or test can be sent instantly to students; they can view them on any type of platform, desktop or mobile: iOS, Android, Mac OS, Widnows, Chromebook. It has functions of grading, evaluation and monitoring the students' school career.

It also contains a teacher training course, accredited and ongoing at CCD Bacău. There are over 5,000 Romanian teachers who use it for free today. (05.05.2020)

Intuitext School - Contains animations, experiments, educational games and tests adapted for the primary cycle. Access is free for teachers and students. Parents have access to a report section where they can track the child's progress. Over 60% of teachers in Romania work with the Teacher's Page section and over 50,000 students in grades I-IV used the existing resources on the platform. (21.05.2020)

MyKoolio - The platform contains over 630 interactive lessons, in the form of interactive educational games, as well as 7,500 math and Romanian language exercises for students in grades I-VIII. The educational content was designed and validated by teachers. It is structured in accordance with the school curriculum in force. The teacher and / or parent have access to reports to see how the tasks are performed by the students.

Teachers can use the MyKoolio platform for free, during the suspension of courses. But solving homework by students on the MyKoolio platform requires a paid subscription. (2.05.2020)

Eduboom - Offers over 1,200 video lessons and more than 800 tests in Mathematics and Romanian Language and Literature for students in grades V-XII. All lessons and tests follow the Romanian curriculum.

Until September 2021, Eduboom.ro offers users in Romania all the necessary lessons for grades V-XII, according to the Romanian curriculum in Mathematics, Romanian Language and Literature, Chemistry, Physics, Biology, Geography.

In Bulgaria, Eduboom is used in over 70% of schools and has over 1 million users. Eduboom currently operates in Bulgaria, Romania, Spain and Italy.(23.07.2021)

Digitaliada - It is a platform with digital educational resources created by teachers in a framework and with well-established procedures. Contains exercises, tests, video tutorials and guides. The Online Learning and Testing section is structured on four levels of access, being dedicated both to principals, teachers from all over the country, regardless of the subject taught, and to

parents and students. (4.05.2020)

Adservio - It is a platform with many administrative functions. It allows the creation of groups and classes, the assignment and correction of homework, the scheduling of control works, the monitoring of students' progress - in essence, it is built as an electronic catalog, with an emphasis on summative assessment. Top-down business model, with promotion through inspectorates (and less a choice of schools). (4.05.2020)

ASQ - It is a platform for "self-learning", with educational content and user-created tests (most, teachers). It is a good solution for lessons created "in an emergency" and for fun exercises. User-created assessment items are used by students for various test runs. (Obs .: Items are neither validated nor calibrated - the degree of confidence in the result is low). (2.05.2020)

Livresq - A platform for creating, editing, publishing, managing e-books, developed through a European-funded project. It can be used to develop lessons and distribute to students. Some of the content developed by other users may be reused. (01.09.2020)

123edu - It is a platform with exercises in mathematics, Romanian language and general culture. It is addressed to primary school students (P-4) and their parents. (4.05.2020)

Kidibot - A platform with gamification learning scenarios, in which children are stimulated to take tests in various fields, entering the "Battles of Knowledge". (4.05.2020)

ExamenulTau - It is an Intuitext product, which covers the gymnasium subject for the National Assessment in the 8th grade. It can be used to prepare for the aptitude test. Each package contains movies, exercises, tests and exam simulations. (2.05.2020)

iziBac - It is a telephone application with which students can practice the baccalaureate program in: mathematics, biology, history, chemistry and socio-human (philosophy and logic). Contains trivia tests in a game-like learning process. You can also simulate baccalaureate exams by solving subjects in a given time. (2.05.2020)

Sabaki Courses - is an online education platform with high school math courses. You learn high school math from anywhere and anytime, through an interactive course and a high-performance online platform. Provides unlimited access to the chosen course. (20.12.2020)

Prepi - An adaptive solution for learning high school math. Make a personalized training plan, especially for the level and pace of the learner, with all the necessary explanations and examples. (2.05.2020)

VreauLa - comes to the aid of students who want to prepare for admissions with a grid exam and offers them a unique learning experience, through constantly updated tests and simulations. Test results are received immediately. He has over 100 tests such as those given upon admission to the Polytechnic University. There are already dozens of tests available for exam preparation for admission to Medicine and Law. (2.05.2020)

DigitalEdu - ideas for learning activities - The platform has a section that includes 900 ideas for learning activities with digital support (of which almost 600 are for distance education) that can be filtered by: curricular area, discipline, class, type of activity (activity in class or homework), individually / in collaboration, digital tool used. The resources are made by teachers and students, after completing a specialized course. All resources are reviewed, filtered and validated by pedagogy specialists. (11.07.2020)

DigitalEdu - examples of activities for distance education - The ideas of activities are published as examples for teachers who do online education during this period. They are elaborated within a collaborative wiki approach made by trainers and participants in the trainings from the CRED Project. (10.07.2020)

Teaching with Europeana - Teaching scenarios that use Europeana's open educational resources. (25.04.2020)

iTeach - Platform dedicated to the continuous professional development of teachers. Includes free online courses, the possibility of publishing in educational journals, news and resources, creating teacher portfolios. Course description is available here: eduVox: Free online courses for teachers. (10.07.2020)

Online-courses: *MOOCs* - The page contains a selection of the best elearning platforms with free online courses offered by prestigious universities or educational institutions. Most courses are in English and cover areas such as communication, psychology, arts, math and science, engineering, economics, management - for personal and professional development.

Each platform

requires the creation of an account to enroll in the courses offered. Usually, online courses that have tutorial support are available for enrollment two to three times a year and last an average of 4-5 weeks. Sometimes a fee is required for the final certificate. Online courses without tutorial support allow immediate enrollment and browsing at your own pace. (5.07.2020)

d) The Ministry of Education approved the operation, starting with the school year 2021 - 2022, of six pilot schools, according to the principles of reforming the education system included in the "Educated Romania" project. Thus, in the six pilot schools, students and teachers will have a teaching-learning-assessment process closer to the wishes and needs of each community and closer to the requirements and demands of a modern, 21st century education, really focused. per student.

The most important component that we consider in the pilotings is the curricular transformation, ie curricular models and framework plans that are adapted by schools according to the needs of the community and the existing human and material resources, which allow the design on areas of learning, not on disciplines, and to focus in a real way on the development of relevant competences, but with strict observance of the curricular objectives specific to each cycle and of the competences from the graduate training profiles. Blended learning teaching-learning models will be piloted, with alternative teaching with physical and online presence, at regular intervals, and new models of narrative assessment, instead of the one based on grades, which will reduce the pressure of assessment on students, especially in the primary cycle.

It will also pilot the replacement of semester theses with a semester portfolio of the student, built on clear assessment criteria presented to each student at the beginning of the semester, an action that aims to assess the real progress during a semester, starting from assessed skills initially and at the end. At the high school level, the reorganization of the common core subject in the 12th grade will be piloted so that, in the second semester, it will allow students to focus on the subjects that are of interest for the baccalaureate and college entrance exams, thus facilitating the student's right to choose certain subjects, which can be later extended to the entire high school cycle.

All the approved piloting initiatives came from state and private pre-university education units, which know best what are the needs and ways to provide the best educational services to their communities. Initially, 20 such proposals were submitted, of which five did not complete the procedure for submitting the documentation, and of the remaining some proposed measures that did not require derogations from the regulations in force or did not consider issues aimed at modernizing education. in Romania.

This step marks the beginning of a profound reform in the field of education in our country. The piloting will be continuously monitored by the General Directorate of Educated Romania, and the models that prove to be successful will be proposed for scaling at national level, after any additional adaptations. The pilot program will also continue in the coming years, to allow a continuous evolution of the education system, which reflects the rapid evolution of the society in which we live. To support this process, the National Recovery and Resilience Plan provides funding for 60 pilot schools in the coming years, including funding to pilot new human resource management models.

In the field of Mathematics Teaching, although there is still much to be explored, today there is a general consensus that technological tools can play a crucial role in the teaching of Mathematics. They, in fact, can be used as mediators in the processes of teaching and learning mathematics.

Interactive Whiteboards in Mathematics

Today, interactive whiteboards (IWB) have become an essential tool and one of the standard pieces of equipment in any classroom at school. As an evolution of the traditional blackboard, IWBs provide opportunities for students and teachers to interact with multimedia content within a classroom setting.

Photo from Samsung

There are different hardware solutions for the realization of IWBs. There are high-cost technologies such as those consisting of large touch-sensitive screens integrating a computer, and low-cost technologies consisting of a computer connected to a video projector that sends images, and cheaper technologies using ordinary personal computers and video projectors integrated with a touch detection system.

However, an interactive whiteboard is still a hardware device, and what makes it a suitable tool for training is the software installed inside which provides the tools and potential required by teaching practice. Some of the functionalities offered by this software are:

- Highlighting: it allows the teacher to show a text and using a brush, underline elements moving the student focus on the concepts of interest.
- Drawing with the finger or electronic pen, choosing colors from the palette, and drawing lines and strokes on the screen, using drawing and image education software for IWB.

- Re-order the sequences of a story by dragging them: one of the exercises proposed to the children to train their ability to recount events in chronological order and to develop the logic of the story is to reorder the images that represent the sequences of a story.
- Listening to the pronunciation of words: The use of audio in a lesson can be used for a variety of activities, one of which is particularly useful when studying languages. You can listen to the pronunciation of foreign words, watch videos of conversations, listen to songs with subtitles.
- Drawing plane and solid geometric figures: In the study of geometry, it is very useful for students in each class to be able to draw figures, model them, observe them in three dimensions and from various points of view. There is software for Lim dedicated to geometry.
- Create concept maps: especially older pupils, who have to study complex and multifaceted subjects, can build thematic maps with links that can be moved, dragged, deleted, and enriched with images, thanks to the interactive surface.
- Interactive quizzes: that can be created by teachers and used during questions.

There is a wide international literature that has investigated the benefits of using IWB in education since its introduction in the school world. The introduction of the IWB has been welcomed by innovative teachers who value its educational potential.

One of the most interesting aspects was that the interactive whiteboard was well suited in the context of teaching all disciplines and mathematics in particular (Beeland,2002; Glover and Miller, 2001).

Erasmus+

Project funded by: Erasmus+ / Key Action 2 - Cooperation for innovation and the exchange of good practices, Strategic Partnerships for school education (European Commission, EACEA)

BECTa

British Educational Communications and Technology agency

The British government agency BECTA (British Educational Communication and Technology Association) in one of their early reviews claims that interactive whiteboards have the potential to enhance teaching and learning through multimedia support and the ability to deliver activities that engage students through their involvement with the tool (BECTA,2004).

Ultimately, IWBs configure a new digital learning space in which the teacher/student relationship can move away from the traditional one-way role and define innovative practices that enhance teaching (Mura, 2011).

OpenBoard

OpenBoard is an open-source, cross-platform interactive whiteboard application capable of running on different operating systems (Linux, Windows, Mac) and designed for use in schools and universities. The first part of the name of the software OpenBoard (Open) comes from the open-source community and therefore the software is freely downloadable and does not require the purchase of any license. OpenBoard was created with the aim of simplicity of use and usability. The main interface is lightweight but offers all the tools needed for teaching activities. The following figure shows the main interface of OpenBoard.

In the main menu, there are some status bars containing: the basic tools (color, lines, eraser), an internal browser, a repository of courses and learning content, etc…

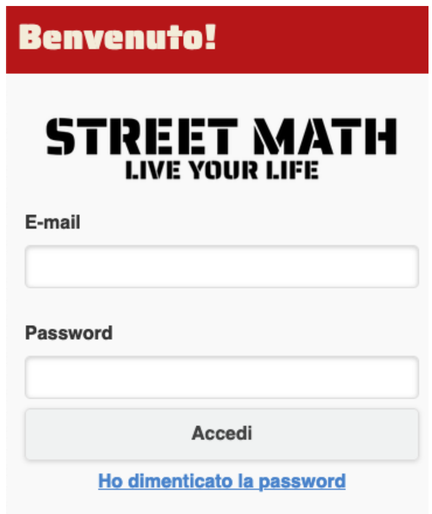
Web App for Teaching and Learning Mathematics

Web apps are applications that can be used through a standard web browser. Thanks to the cloud, web apps are becoming more and more popular and are replacing the old software installed on PCs because of the many advantages they bring:

- web apps do not require a lot of hardware resources (an up-to-date browser is sufficient),
- web apps do not require constant updates,
- web apps can be used by several computers (you will find your information on every computer),
- web apps allow multiple users to interact and collaborate on an activity simultaneously.

Thanks to these special features, web apps are also widely used in education today. Many applications have been developed for the organisation and management of teachers' work, but also many web apps that facilitate teaching and learning in all disciplines.

Within the framework of TEEN (Teenagers Experience Empowerment by Numbers), funded by Politecnico di Milano through the Polisocial Award 2017, Streetmath (https://www.teen.polimi.it/app/index.php)a web app for teaching logic-math concepts was developed.



The teaching activities were built around real scenarios in order to link the concepts studied to concrete situations that could interest young students. The application offers two main modes. The first is a "Challenge" mode in which everyday scenarios are shown and problems are proposed to be solved using mathematical calculations.

The second mode, which can be accessed via the "Maths" button, enables maths reminders, allowing the user to study the operations and basic concepts of each exercise proposed.

The challenge activities propose problems and questions with closed or open answers and are anchored to explanatory images that help the understanding of the text and contain some information useful to answer. The real-life scenarios are those that can be experienced everyday and that can engage a young learner. Some of these scenarios are friends, bus, cooking, trip, telephone, internet, etc... Mathematical problems built around these real-life scenarios will engage students in using mathematical concepts to solve them. The feedback system developed within the application allows for immediate feedback of answers and supports the learning experience by contributing to learner engagement.

From the point of view of good practices in teaching and integrating ICT in the teaching process, we detail below a learning experience carried out thanks to the use of StreetMath.

During the first lockdown due to the Covid emergency (March 2020), a distance learning course was activated, exploiting the pedagogical methodology of the flipped classroom and rewriting it in a virtual way with the integration of Streetmath.

The students were asked to carry out activities on the Streetmath software and at the same time, during a one-hour call per week, explain to teachers and other students what they had learned. Feedback could be requested during the week via Whatsapp from both classmates and teachers. The activity was a success and showed how the use of an ICT tool, supported by the right pedagogical model and planning, can guarantee the delivery of learning paths of quality even at a distance.

Khan Academy

Khan Academy is a platform that allows students to learn online through certain tools such as practice questions, lectures, and videos. Khan Academy allows the study of various subjects ranging from basic mathematics to complex mathematical disciplines, basic geometry, algebra, trigonometry, statistics and probability, calculus (differential, integral and multivariable). Each subject is broken down into individual lessons. For example, Geometry starts with the basics: lines, angles, shapes, triangles, and quadrilaterals, and then introduces more complex topics, such as the Pythagorean theorem, transformations, congruence, similarity, trigonometry, and analytical geometry.

Within the Khan Academy platform, you can find courses and video lessons for students from 8 to 80 years old. The site currently contains over 6000 videos and interactive exercises available on youtube and lasting 10-15 from simple basic maths concepts such as counting to more complicated concepts such as differential equations. The project grew out of the engineering of a model developed by Engineer Khan for his cousin's remedial classes. Today, the platform has been translated into more than 30 languages. The strong point is the possibility given to each student to follow their study path at their own pace. The exercises are also graded and have different levels of difficulty to meet the needs, skills, and abilities of each student.

GeoGebra

GeoGebra is an web application for studying mathematics that lets you, using the various tools offered, draw graphs, study geometry and algebra and solid figures in 3D. GeoGebra can be adapted to different school levels since it is possible to select the levels of difficulty of the proofs to be faced. Therefore, the system can be used from elementary school to university.

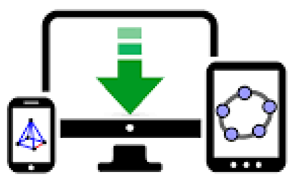
The App is able to support the explanation of study topics and the realization of exercises and tests to verify learning.

GeoGebra is a free application that can be downloaded for both PC and cell phones. It supports iOS, Android, Chromebook and Linux devices, so it is particularly suitable for teaching.

GeoGebra includes several tools for teaching math:

- the graphing calculator: which allows students to create graphs of functions and data, to study equations;
- the 3D calculator: dedicated to the expression of graphs of functions and 3D surfaces;

- Geometry app: deals with circles, angles, transformations and all the elements of basic geometry;
- GeoGebra Classic 6: with calculation tools for geometry, probability, spreadsheets and CAS (the acronym stands for computer algebra system);
- GeoGebra Classic 5: the previous version GeoGebra Classic 6, with similar functions;
- CAS Calculator: to solve equations, develop and decompose expressions, and calculate derivatives and integrals.



Stepping Stones 2.0

It is a framework developed by ORIGO Education to enhance the learning and teaching experience. Stepping Stones 2.0 provides hybrid resources to both students and teachers: they are both digital and printed. This makes the teaching of K-6 math more flexible. The digital side of the framework offers to teachers a set of ready to use slides for every module of the courses that can easily be broadcasted onto the class whiteboard; tools for assessment such as gamified quizzes or digital versions of the printable assessments; tools for monitoring students' progress; interactive material available to students for exercise purpose such as a collection of more than 200 digital serious game tailored to the need of different ages and educational modules.

Fluid Math

FluidMath is an educational technology born out of diverse research efforts. It is designed for classroom, hybrid, and distance learning of mathematics for K-12 and higher education.

FluidMath is an online whiteboard for math. It is a smart piece of mathematical paper that recognizes your handwritten or typed mathematical expressions and with simple gestures.

FluidMath automatically creates graphs, tables, solutions and animations.

Sliders make it easy to visualize connections between mathematical representations and the variables involved.

Animations show connections between abstract mathematical concepts and concrete examples.

A powerful computer algebra system allows you to perform calculations from simple arithmetic and fractions to algebra and matrix operations.

You can also drag and drop PDF and image files into the FluidMath workspace. These become the background against which you can do your math lessons.

FluidMath is seamlessly integrated with Microsoft Teams and is the first collaborative whiteboard designed specifically for math teaching and learning. With this patented technology, teachers and students can synchronously share the same collaborative math workspace.

FluidMath also features patented Mathematics Answerbox technology. Teachers can create FluidMath documents that check the correctness of free-form math answers. This is great for math practice.

And, as an alternative to entering mathematical expressions from a textbox, FluidMath can convert handwritten mathematical expressions into text formulas that can be pasted directly into any text editor.

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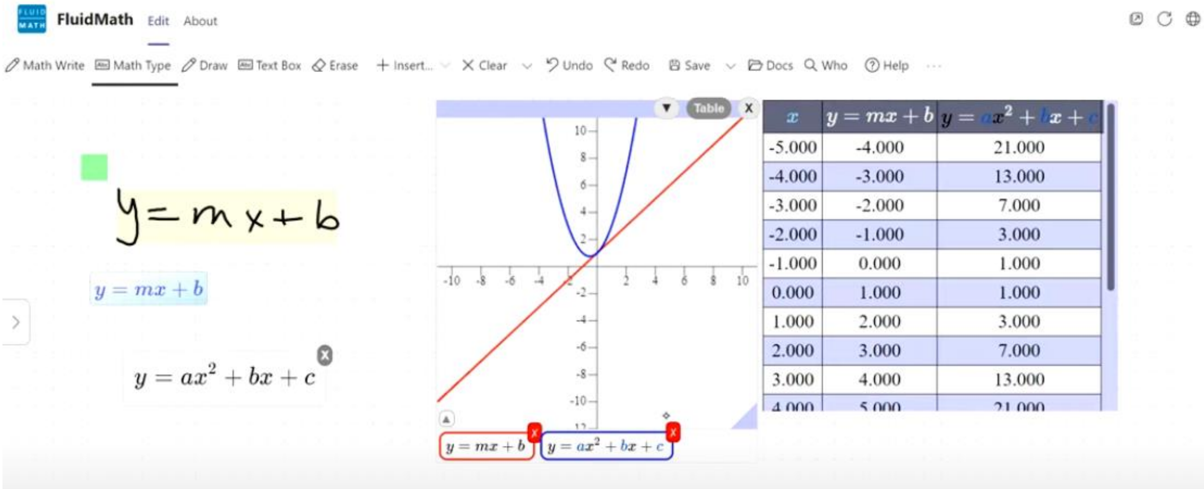
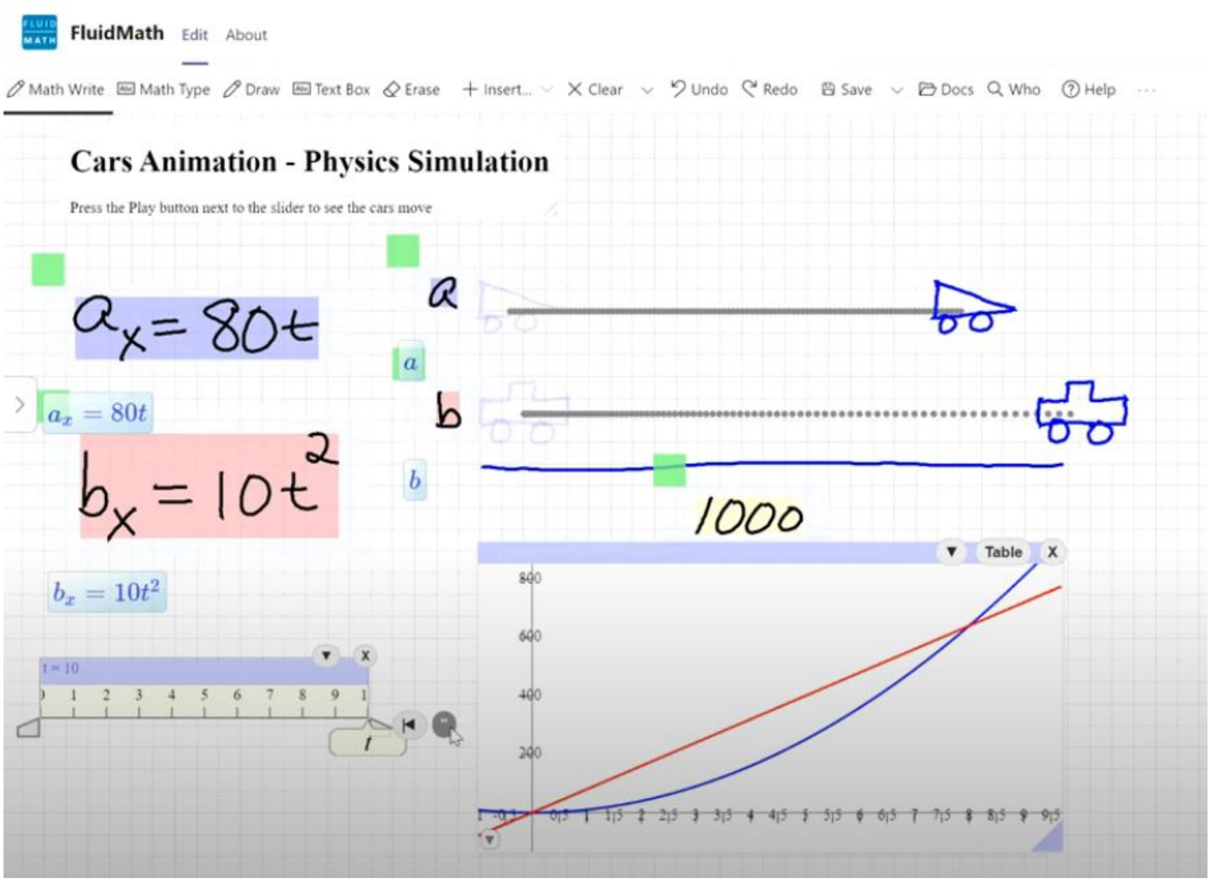




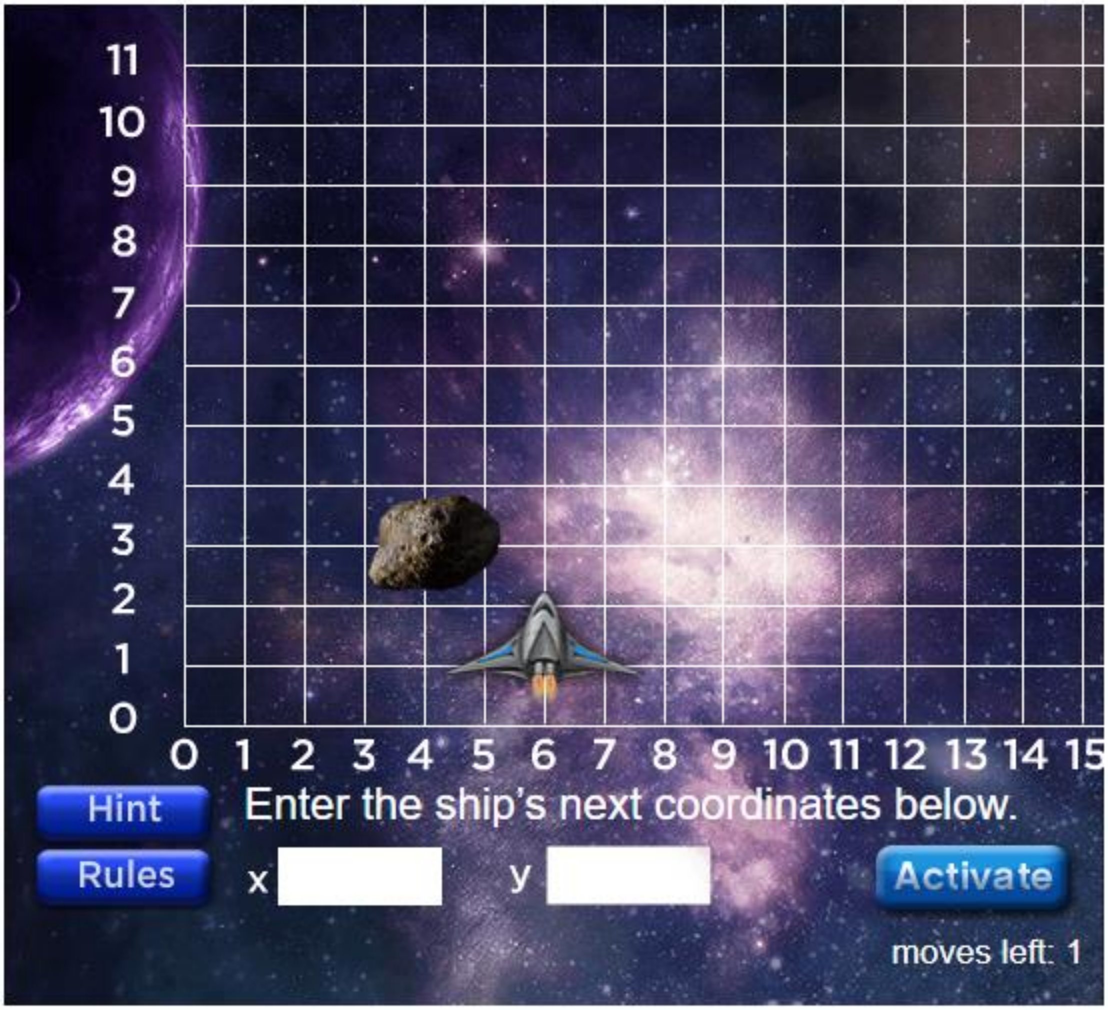
GetTheMath

Get the Math brings together video and web interactivity to help middle and high school students develop algebraic thinking skills to analyze and solve real-world problems.

Project funded by: Erasmus+ / Key Action 2 - Cooperation for innovation and the exchange of good practices, Strategic Partnerships for school education (European Commission, EACEA)

Viewers are challenged through digital challenges they will face using interactive tools provided on the platform's website. Students can further explore the same algebraic concepts through additional interactive challenges on the website. The topics and the challenges inside the GetTheMath platforms are:

- Math in Music
- Math in Fashion
- Math in Video Games
- Math in Restaurants
- Math in Basketball



Mobile app for Teaching and Learning Mathematics

The term mobile learning, often referred to in short as m-learning, is a learning methodology supported exclusively by mobile devices. The main advantages of this discipline lie in having a device, portable or even pocket-sized, from which to receive instruction in multimedia form, complete with text, images, graphics, audio, and video elements or even specially developed games developed ad-hoc for learning. We have seen in the previous paragraphs how mobile devices are now so integrated into the lives of young people. Mobile apps, therefore, play a crucial role in the teaching and activation of formal, non-formal, and informal learning.

Geometry Pad

This is a mobile application. It consists in a virtual graph paper that allows students to draw almost every geometric shape and other geometric features. Students can change the properties of shapes, manipulate them with their fingers, calculate metrics on them, save their work and add notes on the side.

The shapes are displayed on a scrollable and zoomable workbook with a rectangular coordinate system. Specifically, the features that come with the app are:

- Possibility to draw different geometric elements such as points, angles, lines, rays, segments, perpendicular bisectors, tangents, triangles, quadrilaterals, polygons, regular polygons, arcs, sectors, circles, ellipses, parabolas, hyperbolas.

- Tools to create medians, altitudes, and bisectors in a triangle.

- Tools to create special triangles and quadrilaterals: right, isosceles, equilateral, square, rectangle, parallelogram, and rhombus.

- Two additional ways to create an ellipse: by center, end of a major axis and a point on the ellipse; by focal points and a point on the ellipse.

- Compass tool to draw arcs with easily adjustable center and radius.

- Pencil tool for drawing freehand annotations.

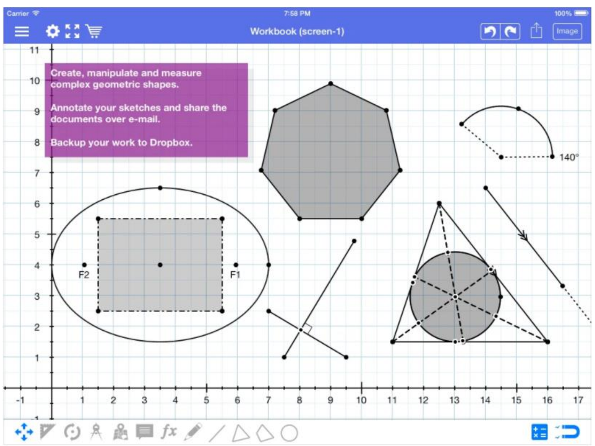
- Text annotations and labels with mixed metrics such as length, angle, perimeter, equation, etc.

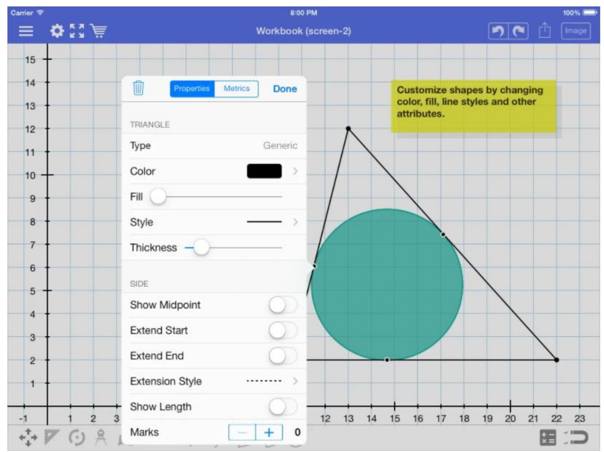
- Transformation tools: rotation, reflection, magnification, translation.

- Create lines and triangles with predefined parameters like the equation of a line and the angles or sides of a triangle.

- Insert images in the document.

- Documents can be saved to your device, iCloud or to Dropbox. Export formats include PDF and image.





Pattern Shapes

Pattern shapes is a mobile application that lets the students understand the properties of shapes and fractions. Students can use the virtual tools to measure angles, change the dimensions and color of forms and annotate text. It is ideal for elementary and middle school students. As they work with the shapes students explore geometric relationships, think about angles, investigate symmetry, and compose and decompose larger shapes.

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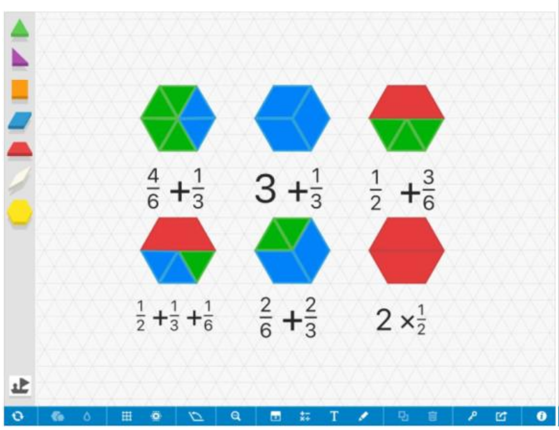
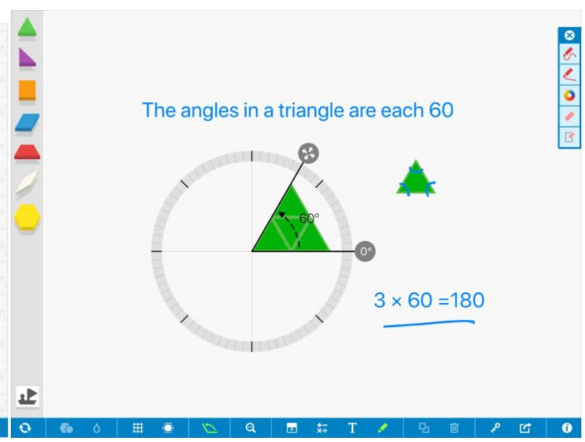
The features provided by Pattern Shapes, specifically, are:

- Rotate the shapes in 15 degree increments.
- Measure angles with the protractor tool.
- Duplicate and change the color and size of shapes.
- Fill in the built-in outlines with shapes or create your own to share with others.
- Choose between a triangular or square grid, or an empty background.
- Hide and reveal work with resizable covers to create your own problems and model strategies.
- Use drawing tools to annotate work and show understanding.
- Add equations, expressions, and descriptions with math text and writing tools.
- Share your work by saving an image or providing a share code to others.

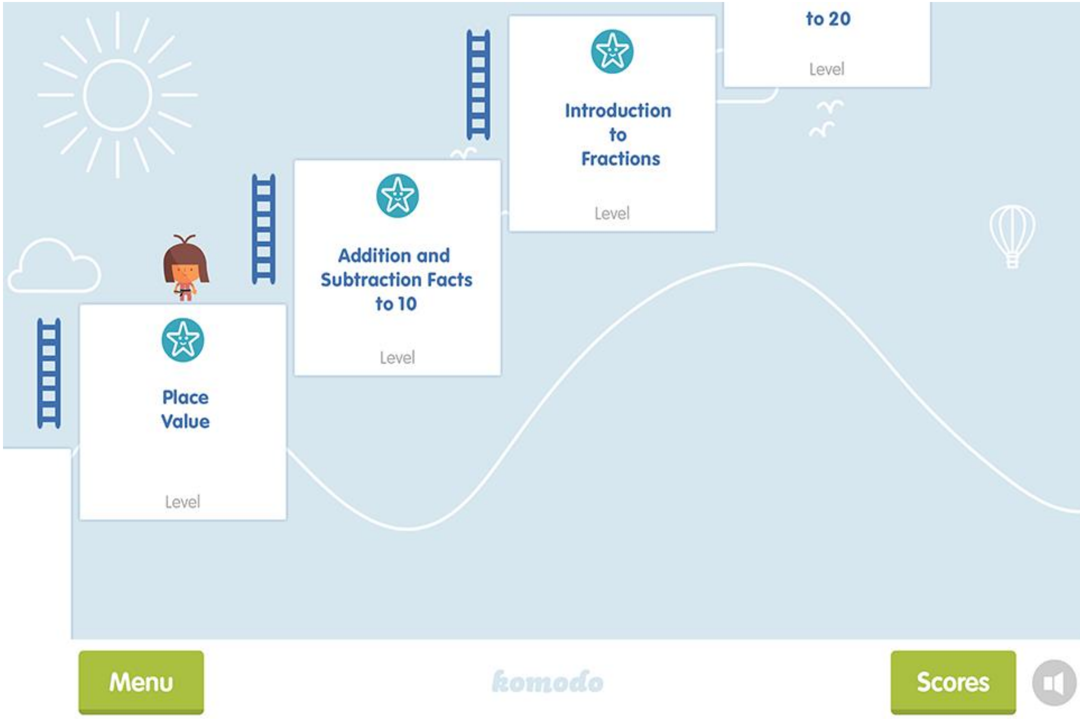




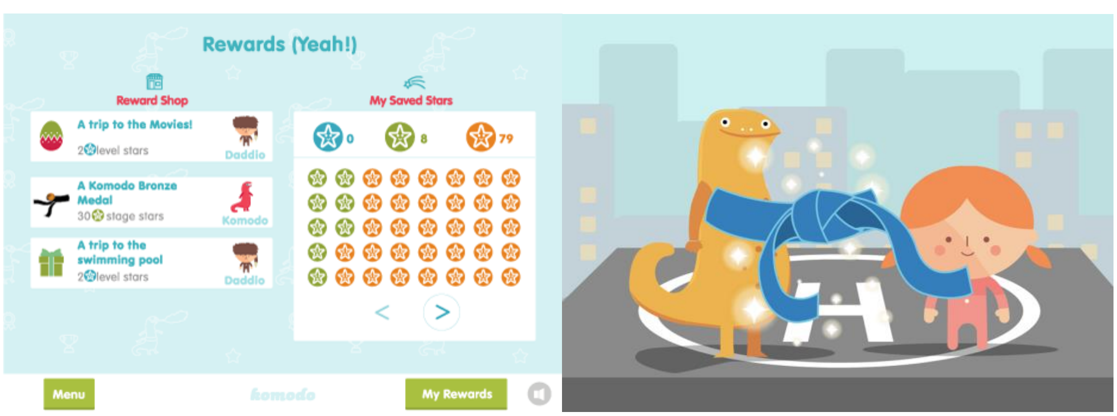
Komodo

Komodo is a multi-platform app developed by teachers to aid the learning and reinforcing of basic maths skills in students ranging from five to eleven years old. It is designed to take short fifteen-minute sessions repeated three to five times a week, making it lightweight and easy to integrate into the children's routine.

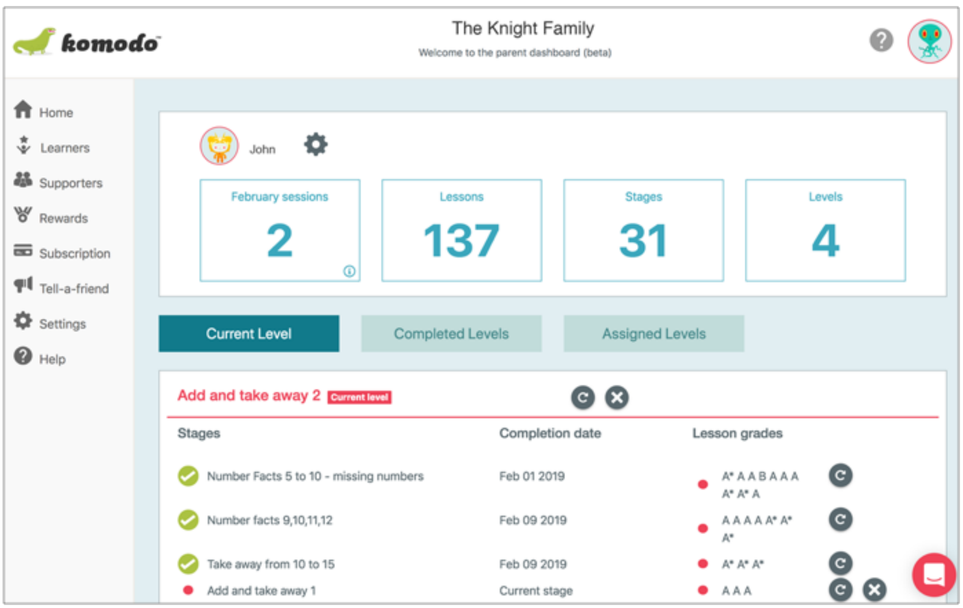
The application offers customised pathways by mathematics teachers tailored to the level of the individual student. After the first access, in fact, a short assessment test identifies the student's level and provides the teacher within the system the tools to provide a personalised work plan. The student can view this pathway and the point within it at which they have arrived.



Activities within the app range from solving simple exercises and quizzes to viewing educational videos on various mathematical topics. As the student learns and completes the exercises, the system rewards him with score and badges that recognise the level he has reached. Upon reaching certain thresholds of knowledge, the player receives a new belt, similar to what happens in sports such as Karate.



An extremely interesting component of this application is the presence of a dashboard for parents. Through this tool it is possible to monitor the player's progress and activities, adjust learning paths and establish real reward paths (such as a physical badge delivered directly from the parent to the student). Through the platform, it is also possible to send messages to students that are displayed directly within the application, facilitating communication when students and parents are not in the same place.



Serious Games for Teaching and Learning Mathematics

Games have always been a useful and effective activity in processes of personal growth and development. In particular, they are man's first form of learning.

Serious games, i.e. games that exploit the playful aspect of the game to facilitate the objectives of education and instruction, aim at education through fun.

The role of the player within a serious game is active, able to interact with the surrounding environment and activate informal learning through an experiential activity. The game allows the user to learn from the experiences he had and to modify his way of acting accordingly.

The playful character of serious games gives space to emotions that make the concepts learned, the knowledge acquired and the skills developed more lasting. The integration of game mechanics such as levels allows users to reuse the competencies and skills acquired in the previous phases, thus consolidating what has been learned. The ability to play with other students enables dynamic competition and collaboration that increases engagement and effectiveness.

Reflexmath

Reflexmath is an application designed to provide individualised and adaptive learning on the basic facts of addition, subtraction, multiplication and division. It is a system designed for students from the second year of primary school, which bases its teaching activities on the use of games.

Players begin their ReflexMath experience at Crabby's Fact Fair with a preliminary assessment of their level of familiarity with basic mathematical concepts. Learning and consolidation of mathematical notions takes place through a series of fast-paced mini-games that stimulate the learner and are designed to provide players with just-in-time feedback regarding the correctness of answers.

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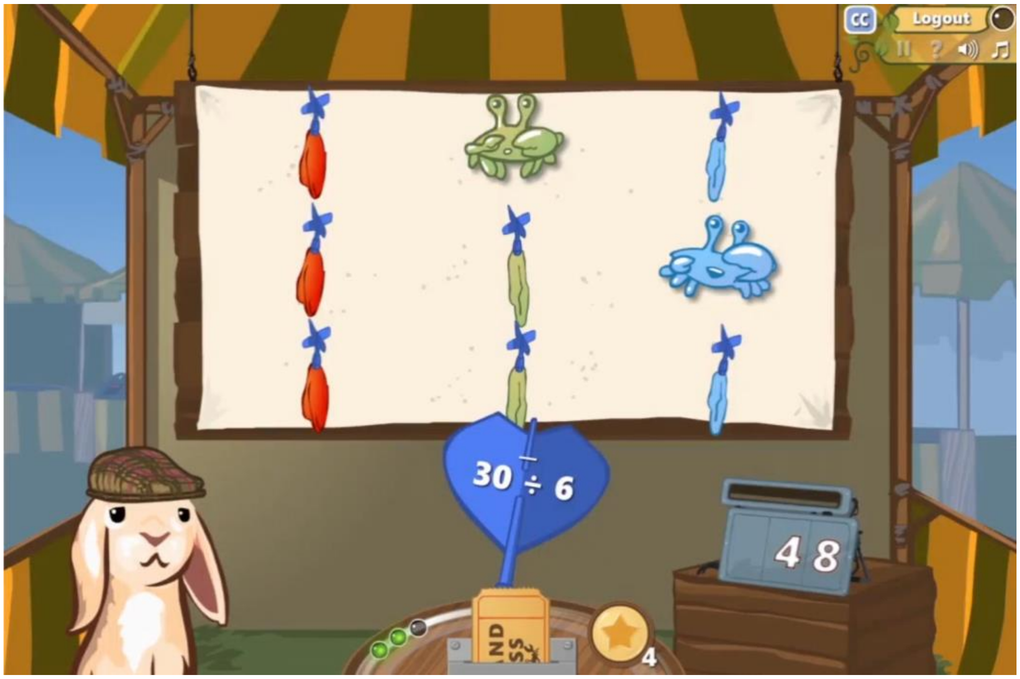
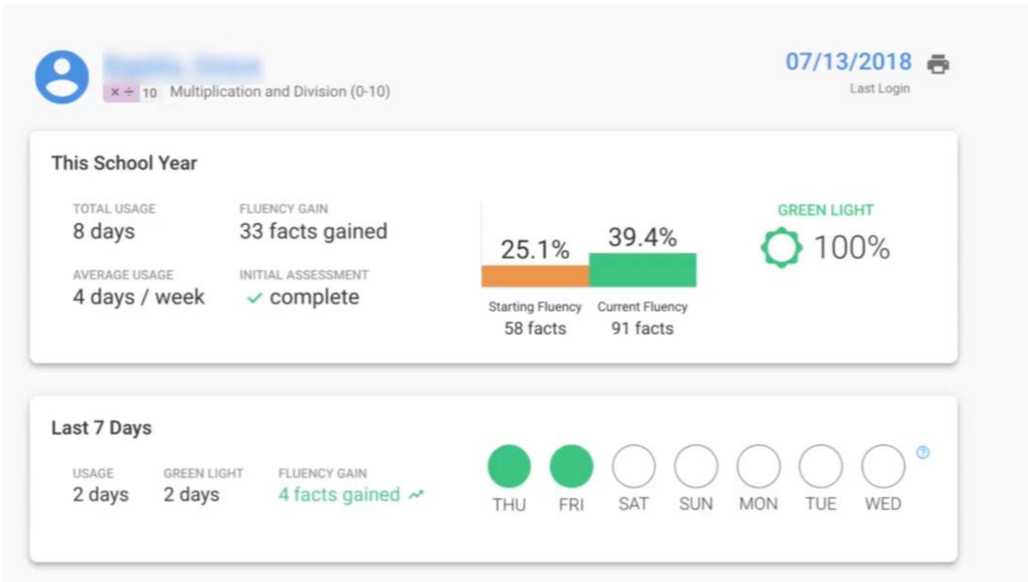


The new concepts are introduced by an in-game avatar (a cute bear called Coach Penny) and the players' progress is rewarded through collectable virtual badges, prizes and points, awarded at the end of each activity based on the players' performance.

The system also includes a dashboard for teachers and school staff to monitor and assess student progress through a simple and effective interface. This dashboard can also be used to print out in-game fluency certificates for distribution to students.

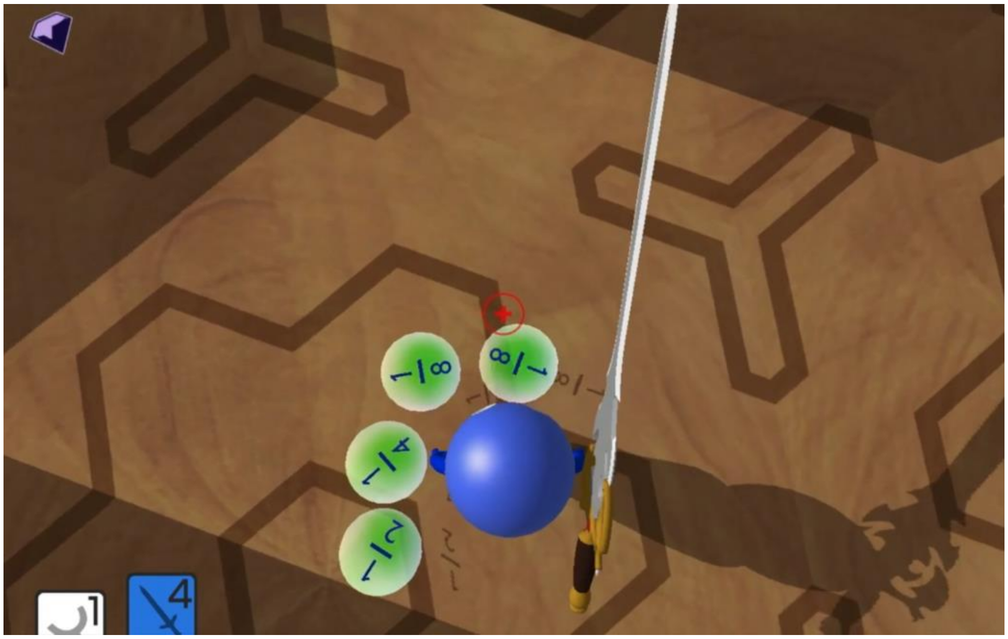


MathBreakers

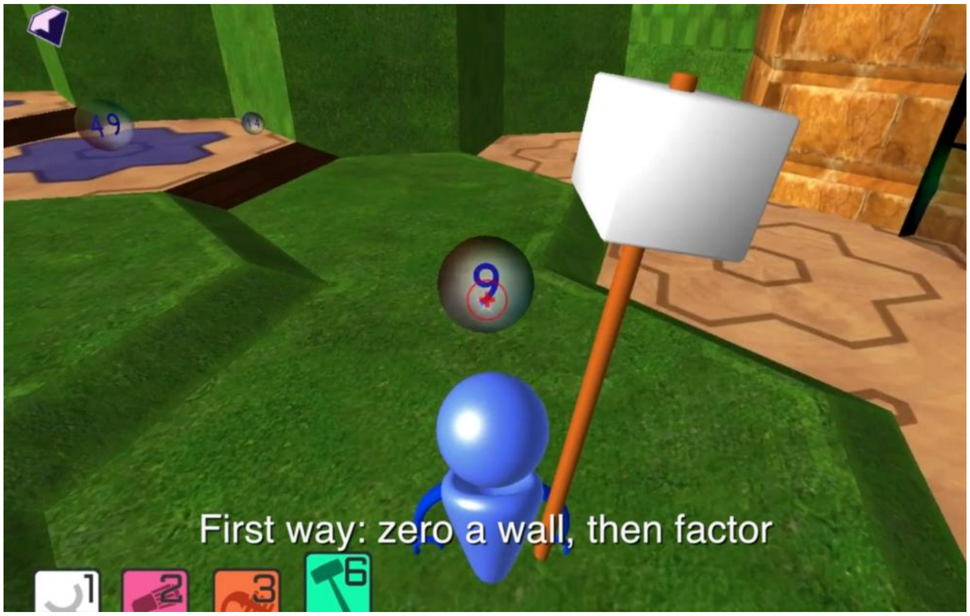
MathBreakers is a 3D action/puzzle game about the basic concepts of mathematics. The game uses an integrated approach that immerses the player in a world where all interactions are based on mathematics. By playing, students learn and consolidate notions related to arithmetic, prime numbers, concepts of fractions, multiples and factoring, negative numbers and concepts of logic.

Learning takes place through a series of challenges. The complexity and difficulty of these varies over time, depending on the level reached by the player: starting with simple challenges, the player will gradually work his/her way through reaching the difficult ones. Within the game, the various enemies are represented by numbers, and to defeat them the player must find a way to zero them out.

To help him in this task, the game provides the student with a whole series of tools that could be described as "mathe-magical": to double a number, for example, the player must use a x2 portal. Similarly, the student can divide a number in half using the splitting-sword, or factor another number using the factoring-hammer.



The variety of tools available also allows players to learn to look at mathematical problems from different points of view: there is no single solution to a given puzzle. Each level is also based on the use of notions learned during previous levels. In this way, players gradually build up their knowledge by continuing to apply basic mathematical principles from start to finish.



Moreover, the game is currently being expanded: a multiplayer mode and a level editor are under development. The multiplayer mode will allow the integration of cooperative/competitive puzzle-solving dynamics that will help boost engagement. The level editor is an extremely interesting feature: the ability to create customised content tailored to the needs of your class provides a high degree of freedom and engagement for teachers.

Math Playground

Math Playground is a portal that collects several math-based video games, divided by target students' age. The games are separated by grade and topic. They go from Kindergarten to 6th grade and some of the covered topics are:

- Algebra Puzzles
- Strategic Multiplication
- Fraction TasksAlgebra Puzzles
- Problem Solving
- 3rd Grade Math
- Visual Math Tools
- Model Word Problems
- Addition Games
- Subtraction Games
- Multiplication Games
- Division Games
- Fraction Games
- Ratio Games
- Prealgebra Games
- Geometry Games
- Fractions

Given the target audience for these games, the graphics are very simple, colourful and attractive. The games allow young learners, in an informal way, to acquire their first mathematical concepts.

DragonBox

DragonBox is a company that developed several mobile apps for learning math and math-related subjects. All of their products are strongly gamified to better address their target age groups which are: from 4 to 9 years old, and from 9 & up. In the field of education, the DragonBox games are one of the most simple (and effective) games, because they offer a really engaging way to include math in children's everyday life. Through interactive games and explanations, young students are introduced to algebra and how variables work. Students have no idea they are actually engaging with educational content, thanks also to the graphics that are colorful and cute.

There are 3 games that are designed to the needs of the first target age group, from 4 to 9 years old. These games are:

- DragonBox Number
- DragonBox Big Numbers
- Dragonbox Algebra 5+

There are also 2 games that are designed to the needs of the second target age group, from 9 & up. These games are:

- Dragonbox Algebra 12+
- Dragonbox Elements

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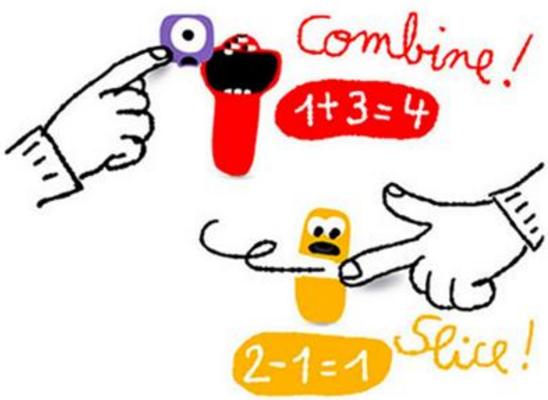
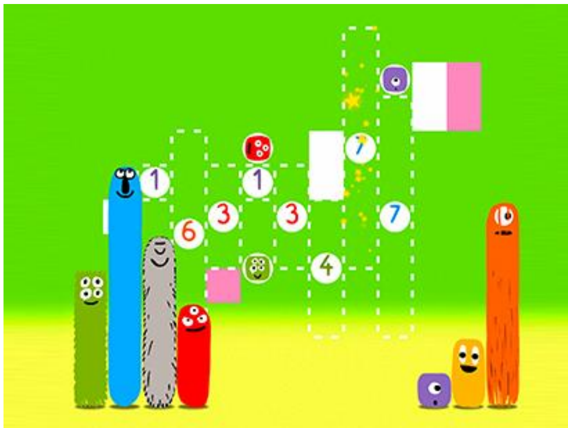
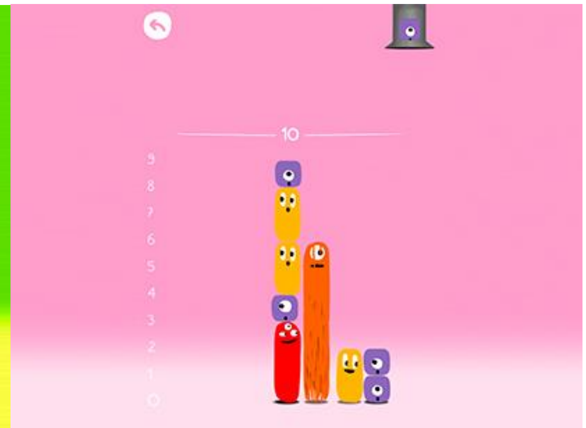
The DragonBox Numbers app is made up of four unique activities that build number sense through game play. Each one focuses on building understanding, flexibility and fluidity with numbers rather than memorizing simple facts.



Here there are little creatures called “Nooms”. Every Noom represents a number from 1 to 10. Players can feed a Noom to another Noom and they combine and stack up vertically. They get additional eyes, and they also change color and make different noises depending on what number they represent. Swipe across a Noom, and it will cut into smaller Nooms.

There are a few modes of play: Puzzles, Ladder and Sandbox. In the Puzzles game mode there are various shapes that have to be filled with Nooms by dragging them. In the Ladder game mode the goal is to create a Noom which is as high as a target to reach. The Sandbox game mode will let the student play with the Nooms on an empty landscape.





DragonBox Big Numbers

Using the DragonBox Big Numbers app the student will jump into the world of Noomia to meet the Nooms and learn big numbers, long addition and subtraction.

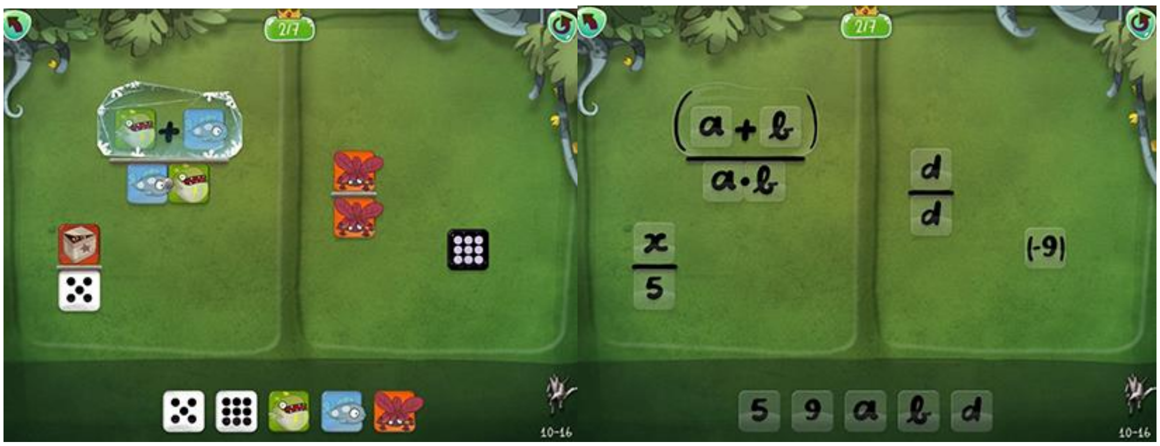
To make progress into the game, the student has to add and subtract resources. Over the course of the game, quantities will get larger and operations will get harder.

DragonBox Algebra 5+

DragonBox Algebra 5+ is an intuitive and engaging game that teaches children as young as 5 the process needed to solve basic linear equations. The student can easily grasp basic algebraic concepts and operations over ten chapters of the game.

An instant feedback mechanism will help to understand what is wrong to gain a sound understanding of algebra.

Without relying on text, it teaches the rules of algebra step-by-step as if the player is really only learning the laws of the universe particular to the game. It uses cards to simulate the algebra variables and with simple gestures the student can simplify fractions, change signum, move an element to the other side of the equals.



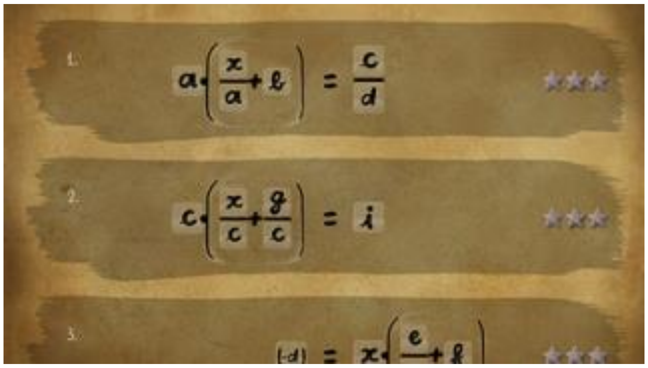
DragonBox Algebra 12+

DragonBox Algebra 12+ is the evolution of the previous game (DragonBox Algebra 5+). It has more levels and more advanced equations. They learn this concept by dragging and dropping objects on two sides of a diagram, reinforcing the idea that what's done to one side of an equation must be done to the other side. The topics covered by this app are:

- Addition
- Division
- Multiplication
- Parentheses
- Positive/Negative Signs
- Addition of Fractions (Common Denominator)
- Collection of Like Terms

- Factorization

- Substitution



Dragon Box Elements

Here, players have to build an army, defeat the evil dragon Osgard and save Euclid’s island. Designed for kids of nine years-of-age and upwards, to manage all that, they have to learn basic geometry and the theorems of Euclid himself. Dragon Box is a good example of a serious game: the students learn math while they have fun with a video game.

Students have to challenge themselves to save Euclid's Island while learning the properties, definitions and relations of geometric shapes through Euclidean proofs.

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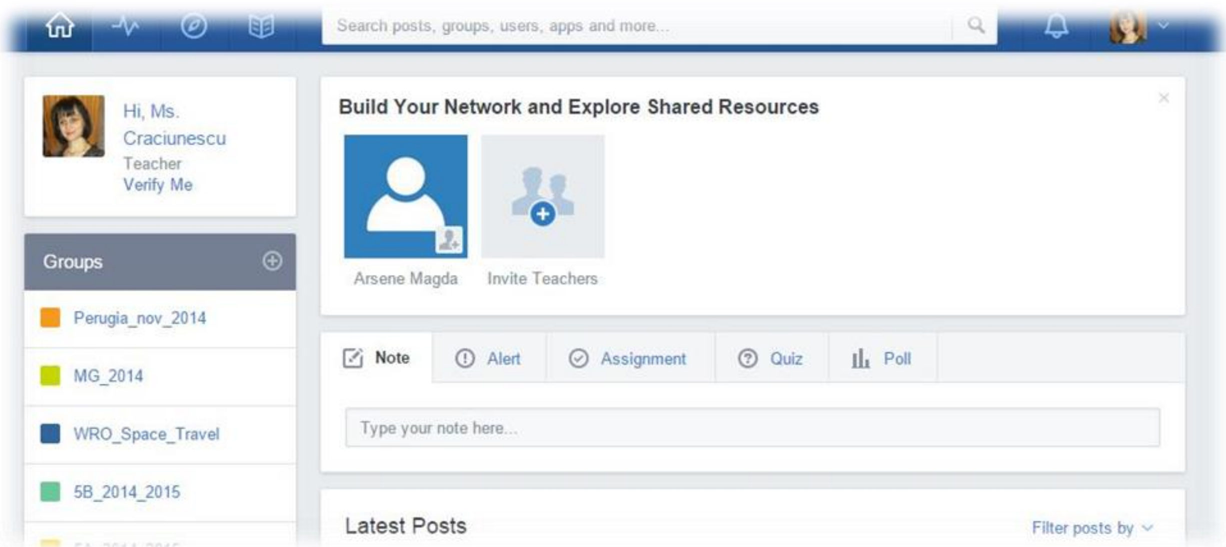
1.3 Materials and Methods

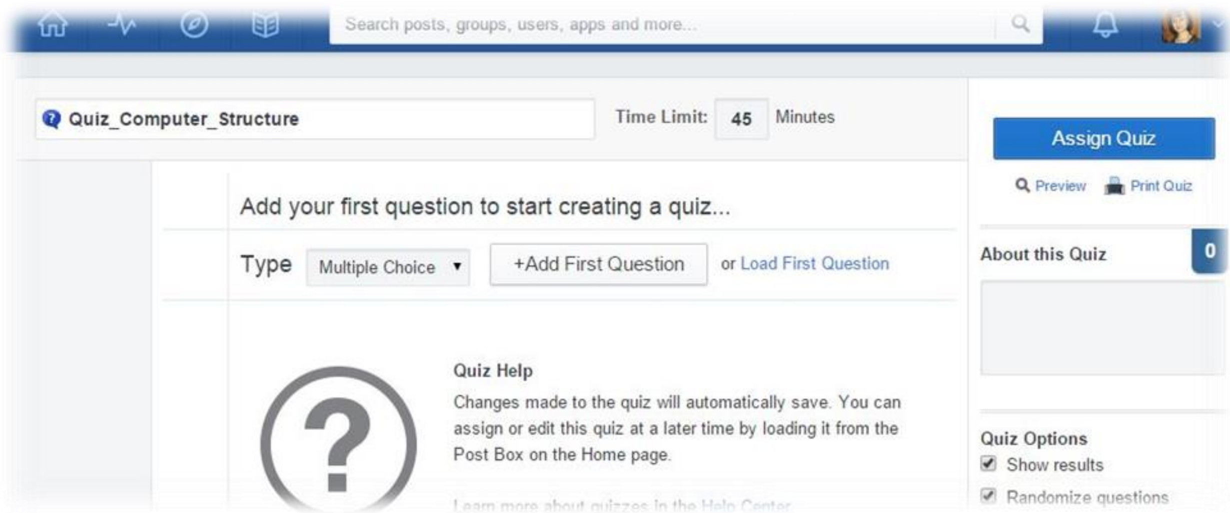
Here are some exemples for organize education procces: *Edmodo* - an eLearning platform.



It is a free eLearning platform for upload all lessons, music, videos and presentations necessary to teaching. It is very easy to use it.

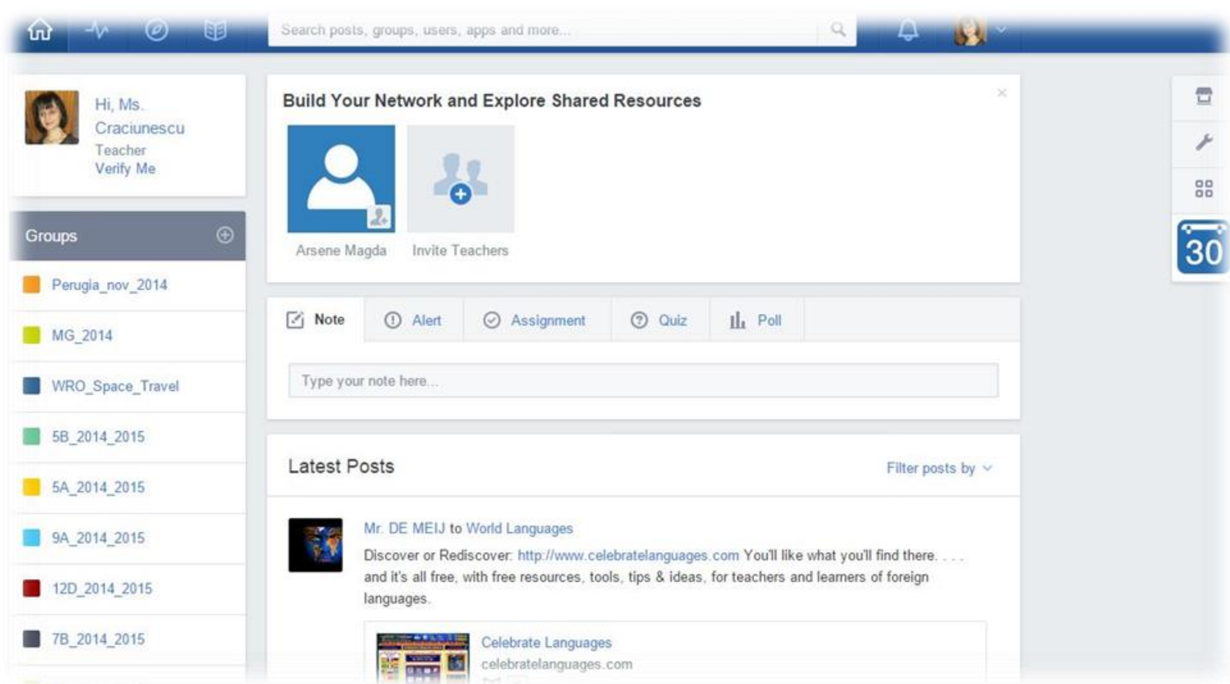
Basically, teacher can manage his class and invite students in class or other teachers to join to the class. We can try this platform by typing this url http://www.edmodo.com

Teacher can have a preview of all students activity and he can manage some quizzes, evaluate, notes and analize them.



Create quiz, alert (if is some urgency), notes, assignment (fixed time and date to hold) and statistics (poll)

For quizzes we have five opportunity: multiple choice, true/false, short answer, ill in blank or matching



The teacher organized the lessons in folders, the contents of folders are in library

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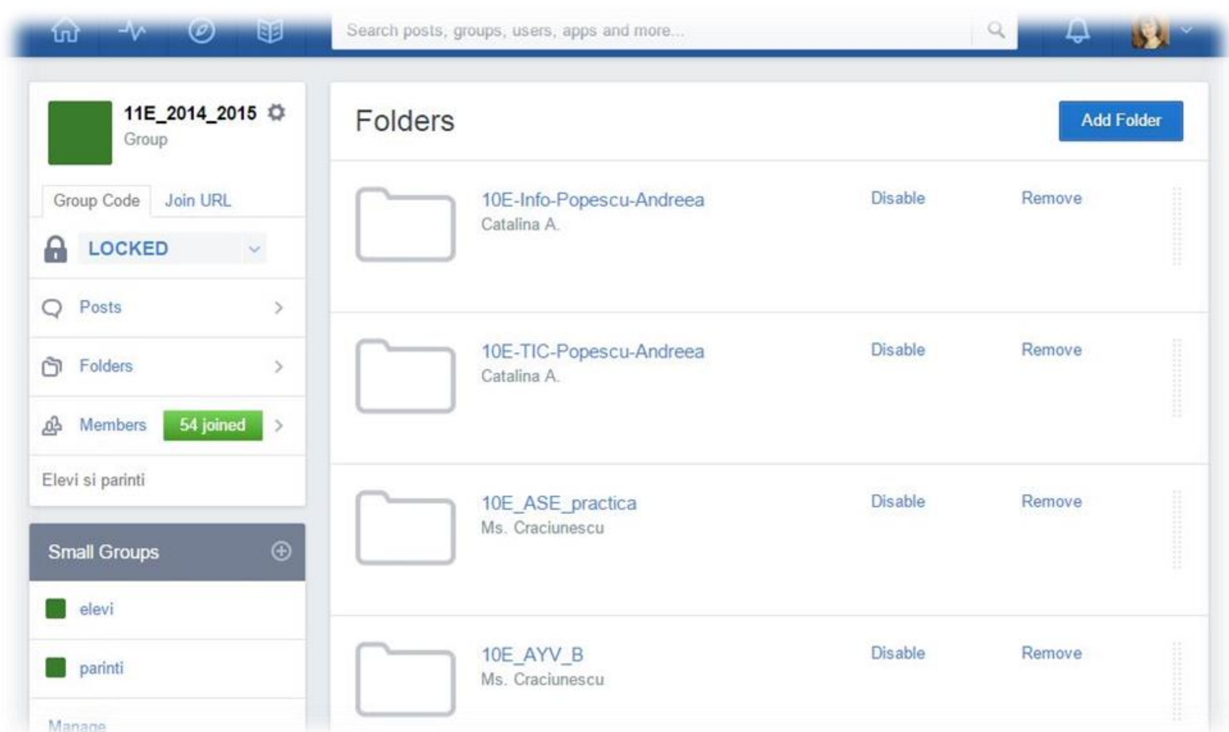
Teacher can give access to parents for to visit their children's activities.

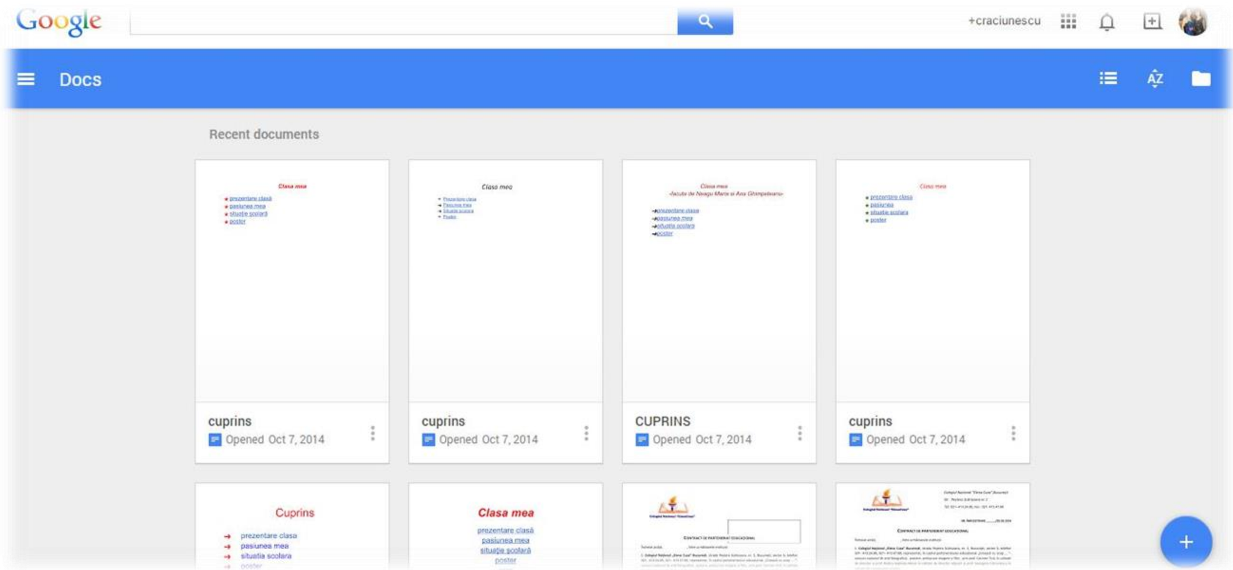


Google Docs:

A suite of software applications available in an online environment, including word processing, spreadsheet, presentation, drawing, and form generating.

Users can access documents stored in their Google account from any computer with an Internet connection. They can create documents in Google Docs or upload and interact with documents created with other applications.

Google Docs are stored within Google Drive. A document's creator, or owner, chooses who can see and edit the document by sending an invitation that shares the document, or makes access to it available, to the new collaborator. We can transfer our work in Edmodo.



Google Drive:

Google’s cloud-based storage system, allows users to store, sync, organize, and share documents from any computer with an Internet connection, but offers no file creation functions. Users have the option to download an application to their computers, and thus be able to sync documents from the cloud to their hard drive. Google Docs are stored within Google Drive.

Hyperlink: An embedded command that allows users to move from one document to another document or website or to another location in the same document. Hyperlinks can be embedded in text, images, icons, or buttons, and are typically activated with a mouse click. Also referred to as a link.

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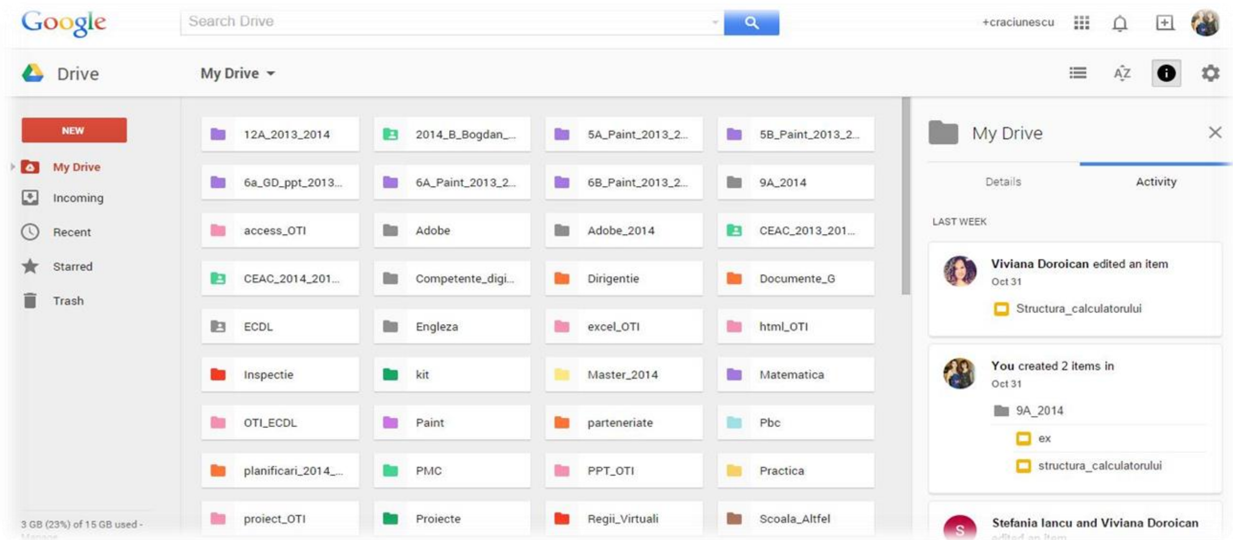
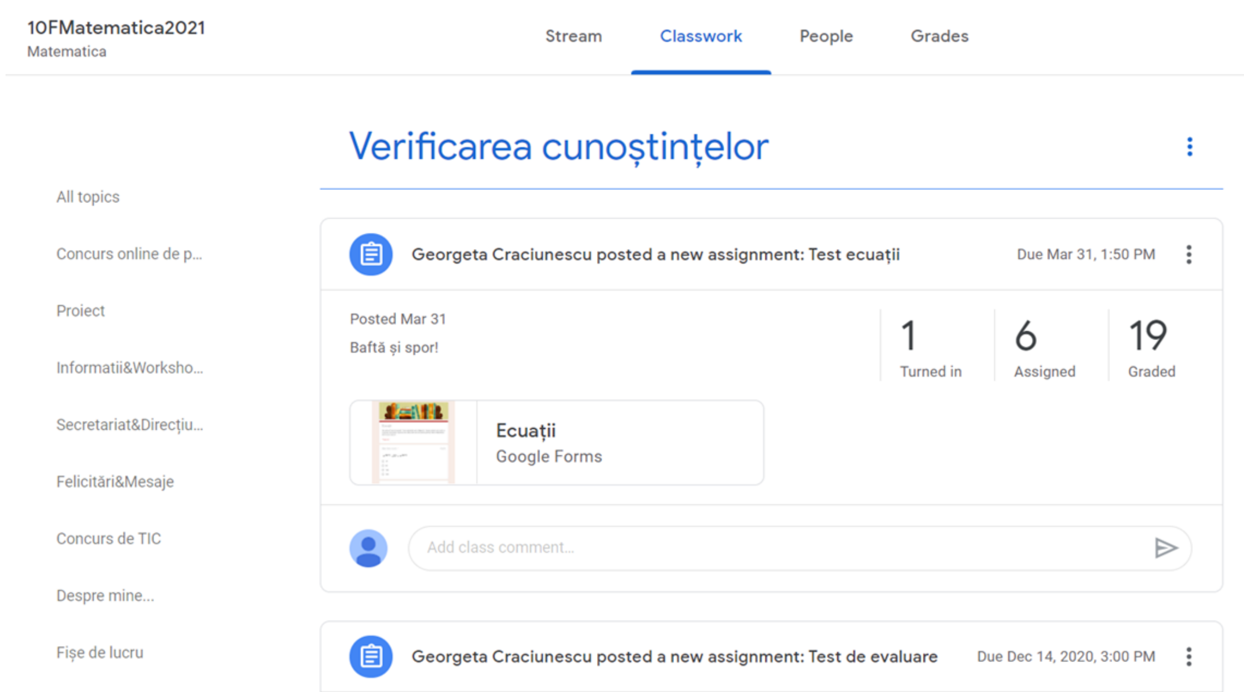
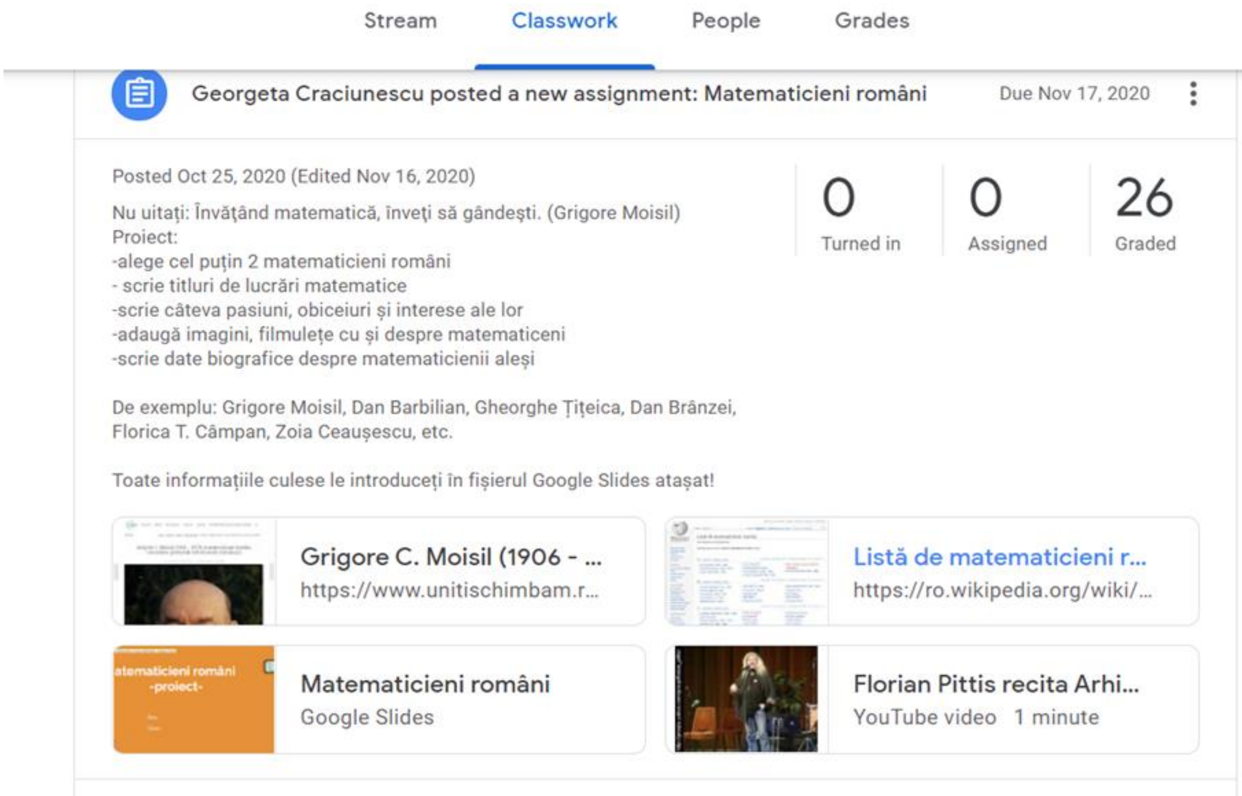


Google classroom:





Geogebra:

the graphing calculator: which allows students to create graphs of functions and data, to study equations;

the 3D calculator: dedicated to the expression of graphs of functions and 3D surfaces;

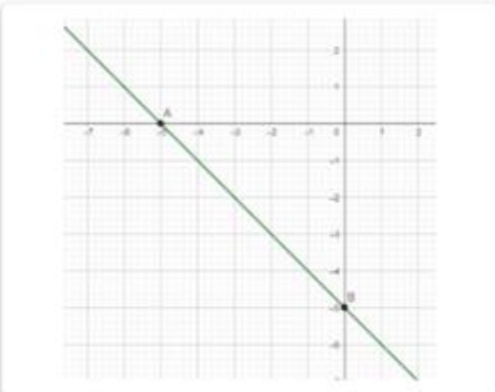
Geometry app: deals with circles, angles, transformations and all the elements of basic geometry;

GeoGebra Classic 6: with calculation tools for geometry, probability, spreadsheets and CAS (the acronym stands for computer algebra system);

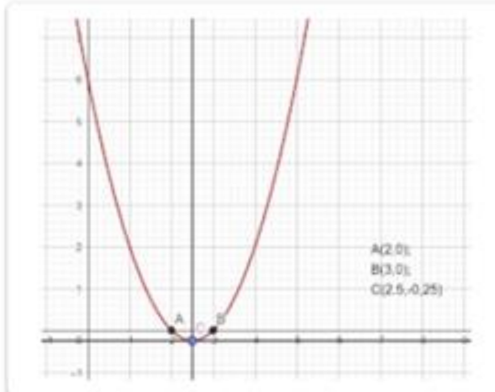
GeoGebra Classic 5: the previous version GeoGebra Classic 6, with similar functions;

CAS Calculator: to solve equations, develop and decompose expressions, and calculate derivatives and integrals.

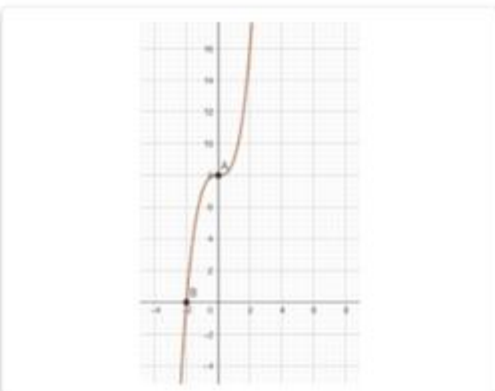
1. Alege imagine unui grafic al funcției de gradul a II-lea. * 5 points



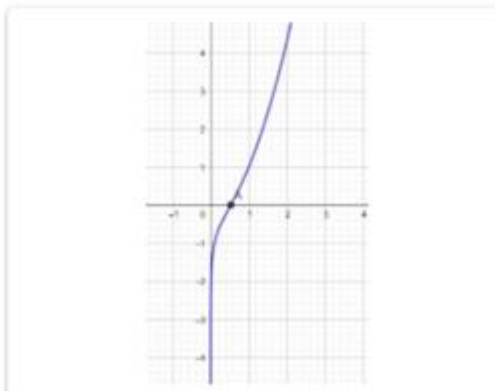
Grafic 1



Grafic 2



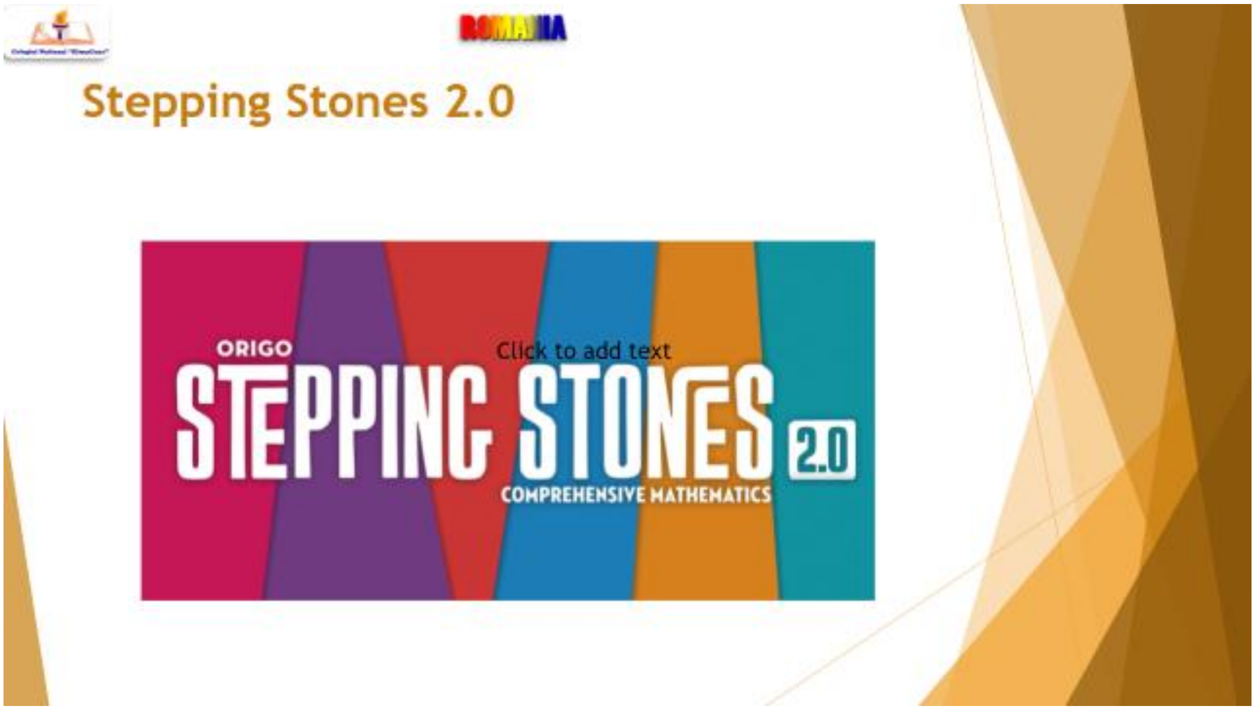
Grafic 4



Grafic 3

Stepping Stones 2.0

It is a framework developed by ORIGO Education to enhance the learning and teaching experience. Stepping Stones 2.0 provides hybrid resources to both students and teachers: they are both digital and printed. This makes the teaching of K-6 math more flexible. The digital side of the framework offers to teachers a set of ready to use slides for every module of the courses that can easily be broadcasted onto the class whiteboard; tools for assessment such as gamified quizzes or digital versions of the printable assessments; tools for monitoring students' progress; interactive material available to students for exercise purpose such as a collection of more than 200 digital serious game tailored to the need of different ages and educational modules.



FluidMath:

is an educational technology born out of diverse research efforts. It is designed for classroom, hybrid, and distance learning of mathematics for K-12 and higher education.

FluidMath is an online whiteboard for math. It is a smart piece of mathematical paper that recognizes your handwritten or typed mathematical expressions

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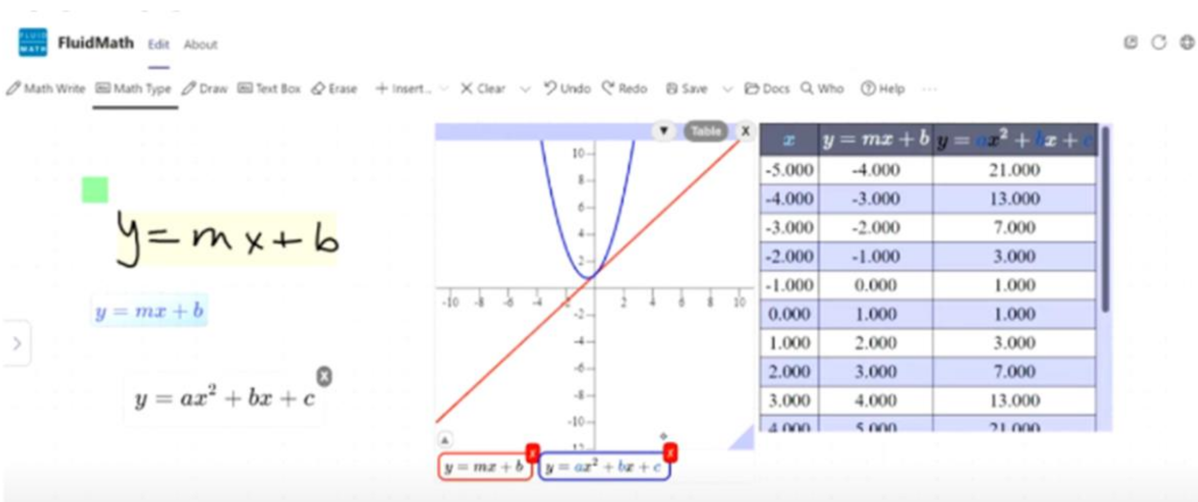

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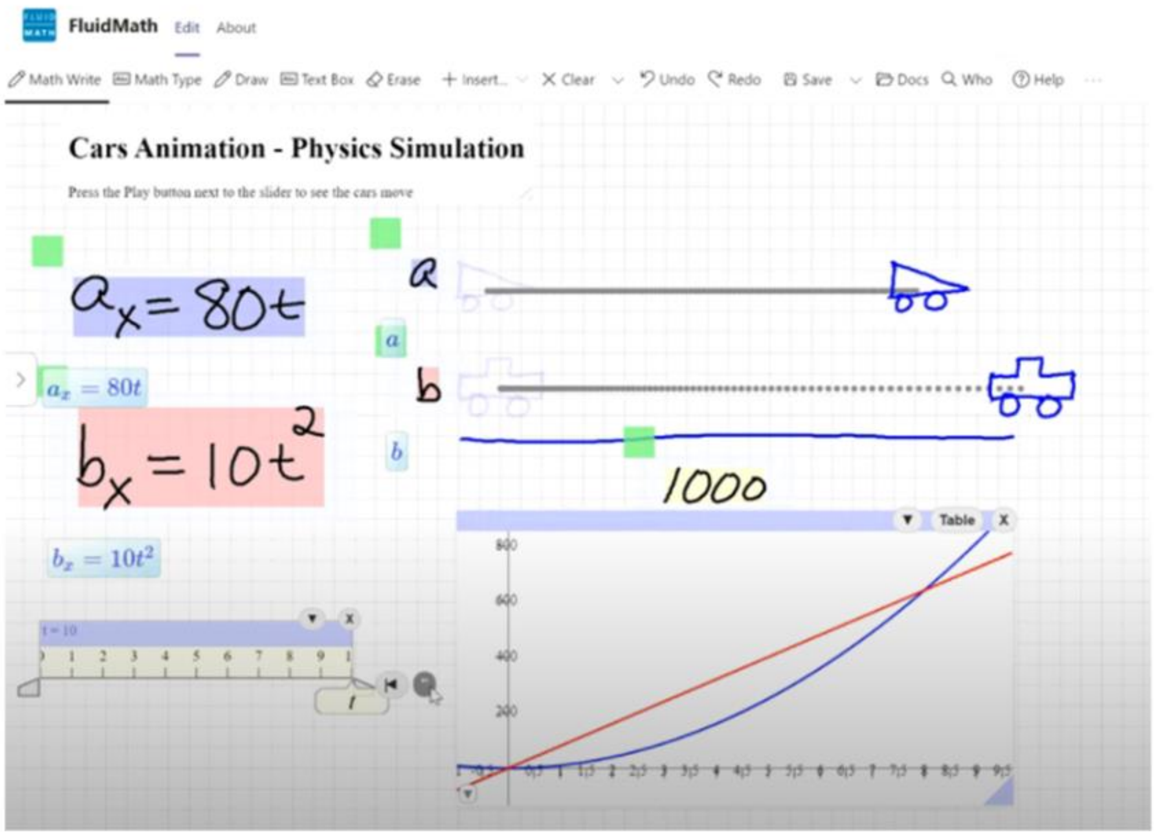

[Dvolver](#) is a very simple tool for creating animated movies. You just select characters and add dialogue. Once the movies are finished you can embed them into blogs or websites or email them to people.

Here's an example Dvolver movie you can watch, [click here]:

Movie Maker:

Turn your videos and photos into movies with **Movie Maker**, a free download in Windows Essentials.

To make a movie with Movie Maker, you'll need some photos and videos on your PC. Import photos and videos from your phone or other device to get started.

Bring your movie to life with a soundtrack that you can edit yourself.

Trim it, split it, speed it up or slow it down: learn more about the different ways you can edit your movies. You can also choose a theme to express your unique style.

Erasmus+

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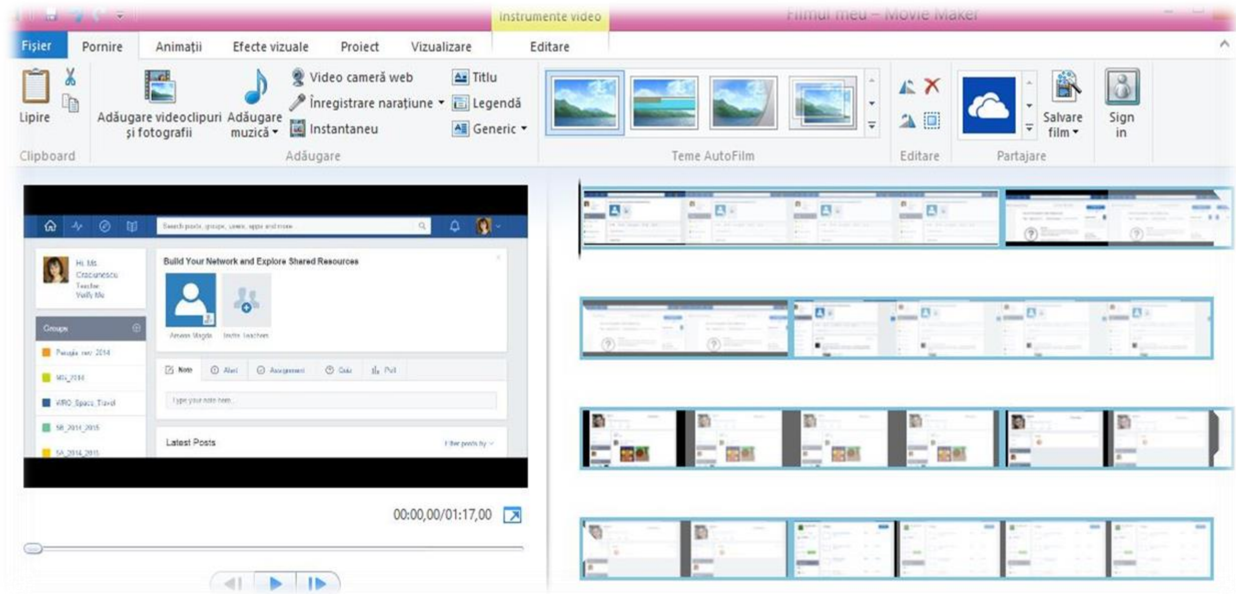


PhotoScape is a fun and easy photo editing software that enables you to fix and enhance photos. Key Features. Viewer: View photos in your folder

Key Features

Viewer: View photos in your folder, create a slideshow

Editor: resizing, brightness and color adjustment, white balance, backlight correction, frames, balloons, mosaic mode, adding text, drawing pictures, cropping, filters, red eye removal, blooming, paint brush, clone stamp, effect brush

Batch editor: Batch edit multiple photos

Page: Merge multiple photos on the page frame to create one final photo

Combine: Attach multiple photos vertically or horizontally to create one final photo

Photoscape

Animated GIF: Use multiple photos to create a final animated photo

Print: Print portrait shots, carte de visites (CDV), passport photos

Splitter: Slice a photo into several pieces

Screen Capture: Capture your screenshot and save it

Color Picker: Zoom in on images, search and pick a color

Rename: Change photo file names in batch mode

Raw Converter: Convert RAW to JPG

Paper Print: Print lined, graph, music and calendar paper

Face Search: Find similar faces on the Internet

Photoshop online - is a free online photo editor with a professional touch. Fix, adjust and filter your images. Manage your images in your browser, no registration necessary.

1.4. Conclusions from each country

Why use technology in science teaching? What opportunities does it offer? There are many reasons for teaching a science subject with the help of technology and the Internet.

Firstly, using technology can be highly motivating for learners. Teachers can engage their students in interesting activities by doing and interacting. Nowadays learners have access to the Internet – at home or at school.

They can play language-learning games or solve cultural quizzes in the class or outside the language classroom using the computer while working at their own pace. With the help of technology students do not only acquire information, but also create interesting materials for their homework and other assignments like web-based projects with sound and colour. Internet-based activities are ideal for all types of learning styles as they provide visual and audio materials as well.

Secondly, the use of technology outside the classroom can offer students great opportunities to become autonomous learners. They can select, analyze, evaluate and use the information collected from the Internet irrespective of the place where they are – in the classroom, at home or on the way to school.

Thirdly, students can become active citizens of the world by collaborating with other students from different countries. They can create blogs, podcasts and wikis about topics of interest. Thus, they can communicate and collaborate online, the ideas and content being generated and created by learners, either individually or collaboratively.

In addition, the use of technology can be time saving for teachers who can prepare and save their lessons on an interactive whiteboard. Thus, they can use all the materials with another class saving teachers the time and expense of photocopying.

The Internet is a useful tool for teachers who want their students to add a dimension of immediacy to a lesson by exposing them to sites such as EDMONDO, WIKI, IEARN, MOODLE or BLOGS. With the help of different sites, teachers can bring new information into the classroom with a view to improving the receptive skills of using technology.

All in all, technology is useful as long as it focuses on the learners` needs. When using ICT tools, teachers must ensure that they improve their teaching techniques and thus make their lessons memorable.

II. Applications of ICT in Mathematics Training

2. Use of digital skills in the field of mathematics

Teaching mathematics occupies a key place in the education of society around the world. Knowledge of mathematics develops logical and analytical thinking, being considered the science that transcends language barriers, as it uses the universal language of numbers and signs, which is understood by all people in the world (Nowak, 2013).

One of the modern methods that is increasingly used both in the teaching of mathematics and other disciplines is Computer Aided Training. Using the computer in math classes makes learning math more attractive, but also allows for better results in students' learning progress.

The use of the computer in the educational process becomes a necessity in the conditions of the accelerated development of the information technology. For the new generations of students, already accustomed to the avalanche of multimedia information, the concept of assisting the computer learning process is an intrinsic requirement.

The potential of information and communication technologies to improve teaching and to make learning more efficient is huge, but it depends on several factors, the teacher's ability to integrate them into teaching, the support and openness of the entire team of teachers and school managers, and technological resources. available.

The school infrastructure provides a significant proportion of access to the computer lab and internet connectivity. The existence of educational software in the school could be a problem as well as the possibility of time or space for training within the school. However, teachers find complementary ways to use the computer, especially since the preparation of lessons takes place mainly in the personal space.

A significant contribution to the integration of ICT in learning is the development of key competences of students, such as transdisciplinarity, positive interdependence, individual responsibility, training and development of social skills. The success of the group depends on the effort made in carrying out the task by all its members, each member of the group assumes responsibility for the task received and communicates with each other to solve the exercises and problems received as a task.

By integrating ICT in learning, we allow students access to scientific knowledge and advanced technologies, data collection for scientific projects and their organization in graphical form, performing simulations and modeling mathematical processes.
Learning mathematics, for example, must be seen both conceptually and as an effective computational technique.

The study of mathematics, more than in any other discipline, does not exist outside the individual work of the student with the pen on paper. As it is considered a very difficult subject by students, countless tools have been developed that are available online in order to facilitate the learning and teaching of mathematics. Existing applications cover all levels of education, from primary school to university. In this module we will present the tools that can help you, so that teaching and learning mathematics becomes much easier and more enjoyable.

Mathematics is one of the oldest sciences, it has a millennial history from which we can provide inspired information that will lead students in an exciting way to the sources of learning the experience of previous generations, information that, it is true, we can find in a click distance. But oral communication has a great flexibility and can offer the teacher's speech, depending on his oratorical qualities, spontaneity and power to adapt to the specifics of the topic.

The teacher can notice the spontaneous reactions of the group of students and can adapt the ways of communicating the information in order to reach the maximum of the proposed objectives. Each teacher is interested in helping students understand the secrets of mathematics, because mathematics is the language with which God wrote the universe (Galileo Galilei).

The presence of information and communication technologies in the lesson is inconceivable today, given the teaching of mathematics, we can not completely give up the classical methods, they proving their effectiveness over time.

The use of the computer in mathematics classes brings an element of novelty in the lessons, by constructing function graphs or by interactively going through some learning sequences. An example of using the computer in math classes is using the Microsoft Excel / Google sheets application used in trigonometry application lessons.

Currently, especially during the COVID-19 pandemic, special importance is given to training skills that allow us to use online tools, resources and online communication. It is important for both the student and the teacher to understand the possibilities offered by ICT and the applications of these technologies.

Teachers should be involved in innovative activities aimed at designing lessons using audio-visual resources, using computers in teaching to develop critical thinking, and European Commission support for the development of this publication does not constitute content approval, which reflects only the opinions of the authors, and the Commission cannot be held liable for any use of the information contained therein. training skills to work with students and improve lessons through innovative methodological design (F.Bereźnicki, 2001).

With the fourth industrial revolution and the beginning of the digital age, an attempt was made to use modern technologies in education, with reference to the introduction of digital textbooks or teaching with the help of electronic learning platforms. However, today's education system is not able to cope with the diversity of students, therefore it does not have the capacity to fully exploit their potential. Teachers, having limited teaching time, cannot customize the learning process of each student in the class, being important for them to benefit from online support in teaching so that they can fulfill this task.

Let us now turn our attention to online tools that can support teachers in teaching. The teacher should study the existing online resources. The Internet provides quick access to knowledge and teaching materials, giving us tools that stimulate the efficiency of teaching-learning. After that, the teacher must choose the means and methods of teaching to help him achieve this goal. For this to be possible, it is important to be aware of the usefulness of a specific tool and to link it to the goal we want to achieve. When you start your search, you

need to be aware of the nature of existing tools on the Internet. The following is an analysis of these tools: Educational applications - applications that allow the teacher and the student to expand their knowledge and develop new skills in specific areas.

You can use the platform to share work tasks / questionnaires / tests created by you, and the students' results will be sent automatically and saved. Getting immediate feedback, the teacher learns about students' knowledge and progress.

Examples of popular educational platforms: Socrative, Padlet, Edmodo, Kahoot, KhanAcademy, ScienceKids etc. Teaching materials creation tools - Tools that allow the presentation of didactic content or the presentation of one's own films, e.g. H5P, edpuzzle, ed.ted, powtoon etc. In the next section we will describe in detail some of these platforms and tools, presenting their advantages and disadvantages.

Every math teacher should become familiar with the tools and remember that the appeal of the message should not overshadow the content they want to convey. The introduction of new and innovative solutions in the classroom does not negatively affect educational success. It is important to keep in mind the friendly atmosphere, establishing a good relationship with the students and the proper organization of the teaching process.

According to experts, the new material is better absorbed when the content is clear and when positive emotions are involved in the learning process. Overall, teachers need to be aware of this progress and implement new solutions in their classroom in order to maintain students' motivation and interest. It should be remembered that the simple description of teaching tools cannot guarantee an increase in the efficiency of the teaching approach, as they only serve to enrich and diversify certain activities of teachers and students.

Interactive knowledge transfer, Student-teacher cooperation in the learning process by facilitating communication. Currently, the teacher should be properly prepared for the use of computer-based methods and technologies in the teaching of mathematics (M.Walak, 2017): The teacher should expand his knowledge of software specific to the teaching of mathematics, the teacher should have skills of use of a computer and administration of a computer laboratory. The teacher should have skills in using the available software and implementing lessons on specific topics, the teacher should gain knowledge of organizing the activity during math classes based on the use of the computer.

The online tools presented below are very useful and easy to use. Their use in math lessons will undoubtedly bring countless benefits to both teachers and students. At the end of the description of the application, there is a link that leads to external pages containing tutorials or movies on its use.

Socrative is a free application. It is a tool with which you can organize online tests in real time using mobile devices with Internet access (smartphones, laptops, tablets or desktops). The application is very easy to use. The platform allows the teacher to monitor the test in real time. By generating a Socrative account, the teacher observes the test and can check the correctness of the answers given by various students. When the test is over, the teacher receives a report on the test results so that he can make a summary of the students' knowledge. Socrative has three main functions:

-Quiz (Test) - the basic function of the application. After generating the account, the teacher has the opportunity to design a test with multiple questions, a test with true / false questions or one with open-ended questions, in which the student can answer questions briefly. The option to add pictures, tables or diagrams to the tasks formulated in the form of text allows to increase the attractiveness of the test for students.

-Space Race - an option that places more emphasis on fun than education. It refers to solving a test against the clock, competing with other students. -Exit Ticket - The teacher's option to ask students questions at the end of the lesson.

Provides teachers with information on the level of understanding of the content of the lesson. Skills and knowledge required Teachers and students using this platform should have basic computer skills. In addition, the website provides clear instructions on how to use it. Benefits for teachers: The test takes place in real time, the test is verified as soon as it has been completed. It reduces the risk of making mistakes in the assessment and saves teachers time. The test results are stored in the Reports section, so the teacher can view them at any time. All tests are conducted online, helping to preserve the environment.

The teacher has the opportunity to analyze the questions that raise the most problems for the students, so that they will know what to correct. Benefits for students: Students get the results as soon as they complete the test, Students are familiar with the technology, which makes this form of testing very attractive to them, Visualizing progress in testing increases student motivation. Disadvantages or limitations: Tasks can be generated in any language, but navigation is only available in English.

H5P is a free application, an open online resource that helps you create interactive and modern content that can be distributed on learning platforms (Moodle, WordPress, Drupal and Tiki). The application allows the writing of content in the form of text or graphics (the material can be enriched with movies or music content), being very useful in modern teaching workshops.

The tool allows: Realization of games and tests (mathematical tests, true / false type tests, filling in spaces, association of images, memory game, questionnaires, tests with multiple choice questions and others). Making presentations (generating diagrams, making cards, multimedia presentations, generating schedules), making interactive movies and virtual tours, making materials with interactive content. The advantage of using the application is that the user does not need to know the languages in which the software is available. It needs a search engine and access to the online learning platform. In addition, many tutorials and movies with instructions for use are available on the Internet.

Benefits for teachers: Content runs on any device, Provides tools and extensive possibilities - thanks to a wide range of exercises that can be created, the teacher has the opportunity to organize lessons at a higher level, H5P can be easily integrated into pages and tools existing website. Benefits for students: Learning through play, solving tests is much more attractive to students than traditional teaching methods, Diversity and interactivity of content. Disadvantages or limitations: Although H5P does not require the user to be familiar with the software, the application editor is much less intuitive than with other such tools, and it is recommended to watch tutorials with instructions designed to make the job easier. Tasks can be edited in various languages, but navigation is in

English.

PADLET is a virtual whiteboard that offers the possibility to collect digital materials in one place. This site allows you to create digital materials on various topics, serving as a place for discussion and helping in the development of "brainstorming" activities, competitions and feedback generation. Making a Padlet is a simple and intuitive process. After logging in we have the possibility to generate a new Padlet board or to use existing schemes. Students can add their posts to the Padlet with comments, questions, feedback, etc. Teachers and students can work on the board in real time. After editing the posts, the teacher has the option to edit the whiteboard settings and share it with other social network users or upload it to a blog or webpage after noting the HTML code. This application can be used on all types of digital devices and computers, being available in 29 languages. Benefits for teachers: Teachers can gather the material in one place, Represents a space for feedback at the end of the class, Interactive presentation of materials, Easy distribution by link.

Benefits for students: The materials are gathered in one place, so they are easy to access by the student, The interactive whiteboard allows the student's attention to be maintained. Skills and knowledge required Both the teacher and the student should have a basic knowledge of computer use. The Padlet application is easy and intuitive to use, without requiring special skills. In the lower right corner there is a "+" sign that allows students / teachers to add a new message. Disadvantages or limitations: The Padlet application requires user registration and, in the basic version, imposes limitations on the number of tables processed, disk space and access by other users. Padlet has an educational version, for which the teacher or school can pay a subscription.

EDMODO is a free educational platform that allows the distribution of educational materials, as well as the use of materials already created by teachers around the world. Edmodo uses dynamic digital tools, which allow the teacher to improve the teaching process. To start using the platform, you need to register by entering your email and password. After logging in, the profile that you can customize is generated. The teacher has the opportunity to create groups or classes and invite other teachers to collaborate. An undeniable advantage of this platform is the ability to interact with external disks such as Google or Office 365, as

well as the fact that the materials are saved in one place - the Edmodo library. Due to this fact, the teacher can easily edit his material, depending on his needs. With the help of Edmodo, the teacher can share information needed by students, such as schedule, assignments and various teaching materials. The platform allows teachers to generate tests, questionnaires and tasks for free, and students to solve them directly on the platform. Moreover, the platform offers the possibility of compiling an electronic diary in which the teacher can keep track of students' grades. Benefits for teachers: The teacher monitors the learning process and the progress of students, The possibility to create projects in partnership with schools in other countries, Easier communication with students, The possibility to form groups, classes, etc., Using Edmodo, the teacher has many data that it can analyze. Benefits for students: Constant contact with the teacher, Viewing the material anytime and anywhere, The student has the opportunity to organize their own file, so that all the material is well organized. Skills and knowledge required Both the teacher and the student must have a basic knowledge of computer use. Edmodo is easy and intuitive to use, without the need for additional skills. Disadvantages or limitations: The platform interface is only available in English.

Kahoot! is a free modern application that can generate and run interactive tests. In order to create and share tests, you need to register, which happens very quickly and easily. The main advantage of this application is the ability to run on mobile devices such as smartphones or tablets, which students have permanently available. Kahoot! it can be used in all disciplines, but also at all educational levels. The teacher can use the applications to recap the previous lesson, to gather opinions about problems, etc. The Kahoot application allows the generation of tests, discussions, surveys or topics for students. The teacher can create Kahoot applications based on text, photos or movies available on YouTube or search for interesting tests using the built-in search engine. Kahoot is very easy to use. After writing the test, the teacher distributes the generated code to the students.After accessing www.kahoot.it, the student enters his name and code, and that's it, this is where the fun and learning begin! Among the questions displayed is a ranking of students, which motivates and encourages them to enter the

Project funded by: Erasmus+ / Key Action 2 - Cooperation for innovation and the exchange of good practices, Strategic Partnerships for school education (European Commission, EACEA)

competition. When answering a question, there is a time limit that the teacher can adjust as he wishes when writing the test. The main advantages of the Kahoot! Application: - Ability to design multiple choice questions, - Ability to add questions and graphical answers, - The test can be sent by email for free, - Participants can solve the test individually, - During lessons, an extensive database of tests can be used, - This application is free, - The number of participants is unlimited. Benefits for teachers: The teacher can perform the test in stress-free conditions, At the end of the test, the teacher can check the students' answers, the ability to download detailed results in Excel. Benefits for students: Using mobile phones, which makes the hours more attractive for students, Students can appreciate how much they liked the test, Students can share their opinions about what they learned with the test, If an answer is provided wrong, the student receives feedback with the correct solution. Necessary skills and knowledge The teacher and the student must have basic computer knowledge. Kahoot app! it is very easy to use, so preparing a test does not require much skill and does not require much time. Disadvantages or limitations: The application interface is only available in English, but the tests can be written in any language. The academy's mission is to provide free educational materials using adaptable modern technologies that identify students' strengths and weaknesses. Teachers can use this platform on a regular basis to help students explore new topics and improve them. Teachers can download reports with which they can identify students who need help. The teacher can also use this application to write students' homework. They can do this using mobile devices and computers. Due to the extensive base of films, articles and exercises available on the platform, the teacher can ask students to review various films or articles before the start of the class, so that they are ready to address the topics from the beginning of the lesson. This is a good way to spend more time on other important activities during the class, such as teamwork or hands-on activities. Another advantage of Khan Academy is that the lessons available on the platform are translated into many languages. Khan Academy makes education easy and enjoyable for young children and students. After class, students can view statistics on their progress, being able to compare them with others in the class. Teachers have access to all necessary data regarding their students. They can download a summary of the class results and find out what topics are causing the students problems. The class profile gives the teacher the opportunity to see the progress of the whole class and to quickly identify students who need special attention in certain directions. Benefits for teachers: Customizes the teaching process during the class, The teacher can observe the students' activity, their progress and the difficulties they encounter and, starting from here, can recommend additional exercises, Facilitates the homework evaluation process. Benefits for students: Immediate feedback in case of wrong answers, Learning through play. Skills and knowledge required The teacher and student should have basic computer skills.

Science Kids is an online platform that contains articles, worksheets, games and tests useful in teaching scientific disciplines. This application gives us countless materials, thanks to which the teacher gets support in the metacognitive process: Using articles, exercises and experiments during class, Using tests to check students' knowledge, The site contains a lot of interesting information, which will help the student to and expand knowledge, The site includes free lesson plans. They can be used during math classes. An additional option of this application is AdaptedMind, which can be used to learn math. The tasks presented in the form of a game are grouped by learning levels, so that the teacher can easily identify the task that interests him.

EDPUZZLE is an application that allows video customization. The operation of this application is very simple: we select our own movie or a movie available in the database and customize it according to our own needs. The tool allows you to add a soundtrack, notes or a questionnaire. An Edpuzzle is a universal tool that can be used by teachers regardless of the subject taught, adding consistent educational value. The use of films in teaching has many benefits: increasing students 'involvement in discussions and sharing impressions, developing imagination by completing watched stories, combining listening and comprehension skills, the ability to test students' knowledge through tests. The teacher who decides to use this application must log in first, then choose the movie he wants to work with. Then, during editing, the teacher can select a soundtrack, questions or comments that can take

the form of open-ended questions. The teacher generates a class on the platform and invites his students to participate by giving them a class code. During the lesson, when the film is presented, the students follow the instructions written by the teacher. Benefits for teachers:

The teacher can check the students' answers at any time and can follow their progress, the teacher can add his own comments to the student's answer. Benefits for students: Represents an interesting form of learning, Get immediate feedback. Skills and knowledge required Working with EDpuzzle is simple and inspiring, without the need for advanced computer skills. However, the teacher should first follow the instructional tutorial to know how to use the platform properly.

Disadvantages or limitations: The application interface is available in English, but the tests can be written in any language. In the current situation generated by the COVID-19 pandemic, most teachers take online courses. The acquisition of knowledge and the formation of mathematical skills is difficult in the absence of direct contact, especially when the student is presented with a new teaching material. Therefore, these tools can be useful not only in classroom teaching, but also in online teaching. The introduction of innovative solutions in education develops creativity, stimulates critical thinking and increases interest, with which students develop their skills.

The latest trends in mathematics didactics emphasize:

- knowledge (facts, spatial forms and quantitative relations, symbolism, phraseology, etc.),
- intellectual proficiency (ability to solve mathematical problems according to the model, training tasks),
- manual skills (ability to work with technical means, tables, graphs, calculator, etc.),
- development of cognitive skills,
- attitudes, students' relationship to mathematics, to learning (Fulier, J., Šedivý, O. 2001. Motivation and creativity in teaching mathematics. Nitra: UKF Nitra Faculty of Natural Sciences. ISBN 80 -8050-445-8).

Due to their special importance in life and during education, logical-cognitive operations (Demkanin, P. et al. 2015. Methodology of creating test tasks and tests. 978-80-89638-28-4).

The word "cognitive" in the broadest sense refers to the knowledge, organization and use of information in the world. Pupil's cognitive qualities include knowledge, skills, and to some extent abilities (Gavora, P. 2011. What are my students like? Pupil's pedagogical diagnostics. Nitra: ENIGMA PUBLISHING s.r.o. ISBN 978-80-8932-91-1).

Cognitive abilities are understood by Kovalčíková and Ropovik (Kovalčíková, I. - Ropovik, I. 2012. Executive functioning as a prerequisite for the ability to learn.

- as one of the most important mechanisms responsible for our knowledge,
- as a set of intellectual skills needed for the effective use of knowledge in selection activities,
- as a concept involving a number of cognitive, mental acts - cognitive operations.

Duchovičová (Duchovičová, J. - Škoda, J. et al. 2013. Psychodidactic conception of the curricular and mediation context of education. Ústí nad Labem: Jan Evangelista Purkyně University. ISBN 978-80-7414-658-9) states that "cognition (cognito), cognitive functions, represent all mental abilities related to thinking and cognition, we dare say that they determine not only the success of the individual in school, but also in life. Cognitive functions are present in all areas of human life, in thinking or mental activities. Among other things, they influence how the student manipulates the presented information, what ideas he / she creates, what and how he / she is stored in his / her memory, what means he / she uses when re-equipping, etc. ".

According to Roth (Duchovičová, J. - Škoda, J. et al. 2013. Psychodidactic conception of the curricular and mediation context of education. Ústí nad Labem: Jan Evangelista Purkyně University. ISBN 978-80-7414-658-9) all intentionally organized activities, which aim to increase the level of existing cognitive functions constitute cognitive education. Cognitive education can operate with different dimensions:

Individuals are presented with tasks that have a general or specific (related to the subject) meaning.

Individuals are encouraged to identify problems and possible solutions themselves, but they have the competence to belong.

Individuals work individually, in small, homogeneous groups, or heterogeneous groups, possibly in classes.

Individuals are guided to develop their cognitive functions through training and learning strategies to reach a solution, or individuals receive information and finally learn how to work with them.

It is clear that "every intervention, implemented in terms of cognitive education, increases the preconditions related to motivation, development of metagognitive processes, intelligence, learning capacity and thus directly contributes to the social inclusion of man and improving his quality of life" (Komora in Duchovičová, J. - Škoda, J. et al., 2013. Psychodidactic conception of the curricular and mediation context of education, Ústí nad Labem: Jan Evangelista Purkyně University (ISBN 978-80-7414-658-9).

An important role of a teacher is to understand students' cognitive process and to support it productively through various strategies and tools. Kalaš together with the team of authors (Kalaš, I. et al. 2013. Transformations of the school in the digital age. Prešov: Polygraf print, spol. S ro ISBN 978-80-10-02409-4) appeals to teachers in the sense that „ among other things, we need to get to know our students, examine their thinking, their perceptions and look for ways to help them learn. ”

2.1 **Regulations or recommendations on teacher-specific digital competences in initial teacher education**

The scientific literature suggests that in order to be able to use ICTs in an innovative way, it is necessary to make use of applications that can support the educational goals emerging from the needs of the knowledge society.

The study of the literature shows also that there are many benefits to using ICT in the educational processes of teaching and learning mathematics in secondary schools. ICT, if properly selected and integrated into the curricula has the potential to transform teaching methodologies by enhancing teacher design work, motivating and improving student and teacher performance, and helping to create a learning environment of collaboration and exchange. Like any innovation, however, the use of ICT at school brings with it challenges and opportunities that need to be analyzed when deciding to activate educational processes based on ICT.

The potential offered by the development of information and communication technologies (ICT) allows a substantial change in teaching methods and learning processes of scientific disciplines and in particular of Mathematics.

Teachers of these disciplines can therefore find new ways for the integration of ICT in the teaching of mathematics in secondary school. secondary. Drent and Meelissen (2008); Ottenbreit-Leftwich, Glazewski, Newby, and Ertmer, (2010); Tsai and Chai (2012); and Wachira and Keengwe (2011) describe two types of barriers that may be hindering the introduction of ICT into the school curriculum, internal and external barriers.

The internal barriers are those that we can trace at the level of the propensity, attitude and training of the teachers themselves, who are often reluctant to make the required changes in their attitude and approach to the study of the subject. It often happens, in fact, that teachers' beliefs related to their own comfort zone, resistance to change, confidence towards ICT and lack of training towards new skills and competences represent the limits of ICT integration in teaching.

As for external barriers, these can be traced to the system level, thus barriers related to both government policies and related to the educational institution such as inadequate resources provided, infrastructure, training and technical support, training and time.

From this it follows that the successful use of technology for the benefit of students depends on the knowledge of teachers and their confidence and competence in the use of technology. Thus, not only must teachers learn to use technology, they also need to learn how to apply it correctly in instructional processes. In addition, they need to know which technologies will be most effective in meeting children's skills, abilities, and needs (Girgin, Kurt, and Odabasi, 2011).

Dhakal's 2018 study investigated which ICT technologies are most widely used in mathematics education and how they are integrated into classroom teaching. It also focused on understanding the attitude teachers had towards these technologies and the barriers they faced in integrating these tools into the school curriculum.

To do this, it administered some questionnaires to survey the percentage of teachers who make use of ICT.

Applications	**Daily (%)**	**Weekly (%)**	**Monthly (%)**	**1 or 2 times a year (%)**	**Not available (%)**	**Never (%)**
Computer	25	12.5	12.5	50	0	0
Projector	12.5	25	25	37.5	0	0
Presentation software	12.5	0	12.5	0	25	50
Graphical applications	12.5	0	12.5	12.5	25	37.5
Word package	12.5	25	0	0	37.5	25
SPSS	0	12.5	25	25	0	37.5
Android mobile apps	50	25	12.5	12.5	0	0

As can be seen from the table, a quarter of the teachers say they use computers daily and about half say they use them at most 1 or 2 times a year. The other rows detail the frequency of use of projector and IWB, presentation software, text editing software, and statistical computing applications. It should be noted that for teachers as well as for young people, the use of apps for smartphones (about half use them daily and only 1 in 8 say they use them once or twice a year) is considerably higher than the use of computers.

Investigating instead the attitude of teachers towards technology in support of mathematics teaching, almost all responded positively.

In mathematics classroom	Very useful and helpful (%)	Useful and helpful (%)	Not so useful and helpful(%)	Not at all(%)
Computer	25	75	0	0
Projector	0	100	0	0
e-mail	25	75	0	0
Internet	37.5	62.5	0	0
SPSS	25	62.5	12.5	0
Mathematica& Geogebra	37.5	62.5	0	0

100% of the respondents answered that technologies such as computers, IWB, Internet, and software for mathematics are very useful and helpful.

Barriers	Major (%)	Minor (%)	Not a barrier (%)
• Not enough access to computers and projectors	25	50	25
• Lack of time in college schedule for projects involving ICT	37.5	37.5	25
• Not enough access to e-mail and Internet	12.5	50	37.5
• Lack of knowledge about how to integrate ICT to enhance curriculum	37.5	37.5	25
• Availability of mathematical soft wares	75	12.5	12.5
• Lack of technical support for ICT tools	62.5	37.5	0
• Students do not have access to the necessary technology at home	87.5	12.5	0
• Teachers do not have access to the necessary technology at home	12.5	50	37.5
• Lack of training, seminar, workshop and talk program in ICT	100	0	0

Finally, when analyzing what teachers consider to be the main barriers, it can be seen that all teachers agree that there is a lack of training in technology. It should also be noted that a fundamental barrier is an impossibility, for students, of using these technologies at home and the lack of specific software for mathematics.

These latter aspects highlight how even today the digital divide is a big problem and how important it is to set up a training course to train teachers and show them the enormous and vast possibility of finding software specifically for mathematics available on the Internet, even in open access mode.

The basic task of the development of cognitive functions in students is to monitor how individual questions, tasks, exercises, problems that the teacher presents in education, develop individual cognitive functions. When a teacher uses, he produces stimuli for all cognitive functions, that is, even the highest ones, so he develops a much wider space of personality, expands his repertoire of answers, and also interferes with non-cognitive functions of personality development. The development of the highest cognitive processes in students creates a pedagogical situation, where the emphasis is also on the teaching of teaching, where it is not only the development of intellect and cognitive structures, but also non-cognitive areas (Portik, 2001).

In the teaching process, the development of cognitive functions is characterized in the set goals of the educational process, which is set by the teacher for students taking into account their (specific) educational needs.

In the history of didactics, several classifications, classifications (taxonomies) of cognitive goals have been developed. The criterion of classification is usually relatively separate areas of psychological activity of students in learning (Turek, I.2014. Didactics. Bratislava: Wolters Kluwer, s.r.o. ISBN 978-80-8138-004-5).

Taxonomies mean arranging cognitive functions into a certain hierarchy according to the complexity of functions, with the higher function automatically including and conditioning the lower function and its application. Taxonomies within cognitivization help the teacher to realize, see into the issue and build tasks and exercises to build the general cognitive functions of the student. Zelina (Zelina, M. 1996. Strategies and methods of child personality development. Bratislava: Iris. ISBN 80-9670-134-7) lists the following axioms of cognitivization and development of cognitive functions:

- it is necessary to develop all cognitive functions, especially the highest - evaluation, synthesis, creativity,
- the development of cognitive functions and processes are based on tasks, exercises, challenges for pupils and students to apply cognitive functions,
- all cognitive functions should be developed in every human activity, in every subject, in every educational activity, all cognitive functions can be developed regardless of the age of the pupil, student, i. both in kindergarten and in adulthood.

2.1.1 Bloom's taxonomy of cognitive goals

The general foray into the intellectual development of the student was prompted by revolutionary research by the American teacher Bloom, who published his findings at the turn of the 50s and 60s (Demkanin, P. et al. 2015. Methodology of creating test tasks and tests. Bratislava: NÚCEM. ISBN 978-80-89638-28-4). Bloom designed a well-known taxonomy of educational goals, which takes into account the complexity of individual cognitive abilities and their complexity and helps to focus on learning even more demanding thought operations.

Benjamin Bloom divided thinking into six levels: memorization, understanding, application, analysis, synthesis, evaluation (Hunter, M. 1999. Effective teaching in a nutshell. We increase the effectiveness of teaching in all types of schools. Prague: Portal. ISBN: 80 -7178-220 -3).

The original Bloom's Taxonomy of Learning Objectives was published in 1956 and was created as a tool for classifying test tasks. While the original taxonomy until recently influenced the creation of educational programs, especially in the Western world, the revised Bloom's taxonomy adapted for teaching by LW Anderson and DR Kratwohl (collaborator of BS Bloom) has now progressively influenced the creation of educational programs throughout the modern world, including in our country (Demkanin, P. et al 2015. Methodology of creating test tasks and tests Bratislava: NÚCEM (ISBN 978-80-89638-28-4).

Compared to the original version, the revised Bloom's taxonomy is characterized by three key changes:

- the revised Bloom's taxonomy is two-dimensional,
- moving the last two levels in the dimension of cognitive processes,
- terminological adjustment, when the original names of individual levels were reformulated into activity verbs, which made the revised Bloom's taxonomy more consistent with the way of defining questions (Valent, M. 2007. Taxonomy of educational goals in the new dress. Pedagogical perspectives. Vol. 16, no. 5 , pp. 14-16 (ISSN 1335-0404).

Figure 1 shows the relationship between the original and revised Bloom's taxonomy of cognitive functions

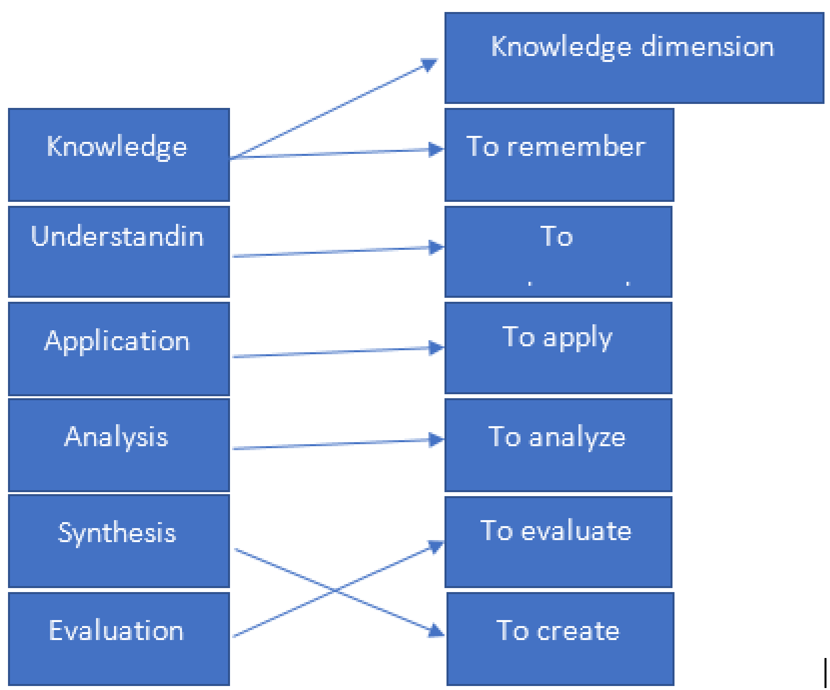


Figure 1 Original and revised Bloom's taxonomy (Anderson, L.W. - KrathWohl, D.R., 2001, p. 268) In: Valent, 2007, p.15)

Project funded by: Erasmus+ / Key Action 2 - Cooperation for innovation and the exchange of good practices, Strategic Partnerships for school education (European Commission, EACEA)

The following classification table (Table 1) makes it easier for teachers to distinguish between levels of acquisition of knowledge (knowledge), helps in the planning of teaching, its implementation and evaluation of its results, in the creation of questions and tasks. One dimension consists of types of knowledge - they are expressed by nouns that the student is to learn. The second dimension consists of cognitive processes, arranged in a hierarchy (they are expressed by verbs that carry different levels of demands on the student's learning). It starts with the least demanding thinking processes (simple memory management) and ends with the most demanding processes that develop the student's creativity.

Classification table for distinguishing the level of acquisition of knowledge

			Cognitive processes dimension					
			1	2	3	4	5	6
			To remember	To understand	To apply	To analyse	To evaluate	To create
Knowledge dimension	A	factual knowledge						
	B	conceptual knowledge						
	C	Procedural knowledge						
	D	Metacognitive knowledge						

According to Blašek (Blaško, M. 2016. Quality in the system of modern teaching [online]. [Cited 28.9.2016]. Available at: http: //web.tuke.sk/kip/main.php?

Om = 1300 & res = low &

menu = 1310) the student has to master in a specific subject within the educational process:

- in addition to the facts (A - factual knowledge: basic information with which the student must become acquainted),
- and the relationships between them (B - conceptual knowledge: information on the relationships between elements within larger structures),
- also techniques and methods (C - procedural knowledge: information on procedures, methods, algorithms, use of skills),
- thinking and problem-solving strategies, learning to learn, being aware of one's own possibilities and abilities (D-metacognitive knowledge: information about cognists, but also about self-awareness).

The dimension of knowledge is closely related to the dimension of 6 levels of cognitive abilities. The student must have certain knowledge or be able to find certain information in order to be able to apply, evaluate, create, understand something. Blaško (2016) describes the individual levels of cognitive processes as follows:

- Ability to remember: is the ability to acquire, store and recall knowledge from long-term memory. Memorization is the knowledge of information. Objectives at this level require the student to recall, recognize, or reproduce data he or she has previously learned.

When setting goals at this level, activity verbs are most often used: state, define, name, name, define, pronounce, write, describe, break down, get to know, repeat, reproduce, supplement, assign, select, determine, etc.

- Ability to understand: is the ability to construct the meaning of the content of information presented in verbal, pictorial, or symbolic form (graphically). Goals at this level require the student to be able to explain the essence of what he has learned in his own words.

Typical business verbs: otherwise formulate, prove, illustrate, explain, characterize, illustrate, interpret, interpret, estimate, translate, translate, calculate, express in your own words, draw, sketch, sketch, fill in, distinguish, compare, derive, check, use , repair, expand, etc.

- Ability to apply: is the ability to use or carry out a procedure in a given situation. Pupils are expected to have the knowledge mastered at previous levels of memorization and comprehension. Application-level goals require students to be able to apply the learned knowledge directly in a particular example, to apply it in a specific situation.

This ability requires meaningful use of abstractions and generalizations (theories, laws, principles, relationships, methods, procedures, concepts, rules) in specific situations. Typical business verbs: apply, demonstrate, illustrate, design, interpret data, solve, quantify, test, calculate, search, sketch, plan, use, prove, orient, organize, discuss, document, document, etc.

- Ability to analyze: is the ability to break down material into its components, to determine the interrelationship of these parts and their relationship to the overall structure or purpose. Goals go beyond memorization and the application of knowledge. They assume to create deductions or hypotheses in the student on the basis of research.

At this level of cognitive goals it is necessary to make an analysis of complex information (system, process) into individual elements, steps, parts. It is a justification of the steps of the procedure, specifying the conditions. Typical business verbs: analyze, justify, analyze, distinguish, specify, subdivide, classify, parse, determine, specify, and so on.

- Ability to evaluate: is the ability to assess knowledge and information on the basis of criteria and standards. The goals are to determine the value of the product on the basis of learned standards and criteria.

Typical business verbs: argue, defend, decide, oppose, support (opinion), compare, assess, evaluate, verify, justify, state advantages and disadvantages, appreciate, criticize, take a stand, state pros and cons, compare with the norm and below .

- Ability to create: is the ability to bring elements (information) together, to form a coherent or functional whole, or to reorganize the elements into a new whole or into a new structure (it can be a plan, report, solution procedure). The goals are to be able to create something new, your product, original product, mental or physical performance.

Typical activity verbs: categorize, classify, combine, modify, write an essay, create a portfolio, propose a solution, organize, reorganize, plan, summarize, draw conclusions, create, develop, propose a new (procedure), etc.

2.1.2 Niemierk's taxonomy of cognitive goals

Polish pedagogue Niemierko (Blaško, M. 2016. Quality in the system of modern teaching [online]. [Cited 28.9.2016]. Available at: http: //web.tuke.sk/kip/main.php? Om = 1300 & res = low & menu = 1310) recognizes 4 levels of cognitive goals:

- memorization of knowledge (equipment, re-knowledge, reproduction of terms, facts, concepts, relationships, classifications, laws, theories, procedures, operating principles),
- understanding knowledge (present memorized knowledge in another form, summarize, organize, tell the content in your own words, explain the meaning of quantities in relationships),
- specific transfer (use of knowledge in typical situations - solving typically school tasks, applying knowledge according to the submitted model, solving similar examples),
- non-specific transfer (use of knowledge in problem situations, perform analysis and synthesis, formulate a plan of activities, evaluate, solve problem tasks).

In contrast to

Bloom's taxonomy, Niemierk's taxonomy summarizes all the above logical-cognitive processes into a single category, and that is non-specific transfer.

Education throughout the modern world has finally moved from knowledge to competence, and the taxonomy of goals helps us see where to go. The ability to think creatively, solve problems and make the right and useful decisions is a basic goal of education. In order to help students achieve this goal, we must sensitively distinguish between higher levels of thinking and learn to evoke them in students.

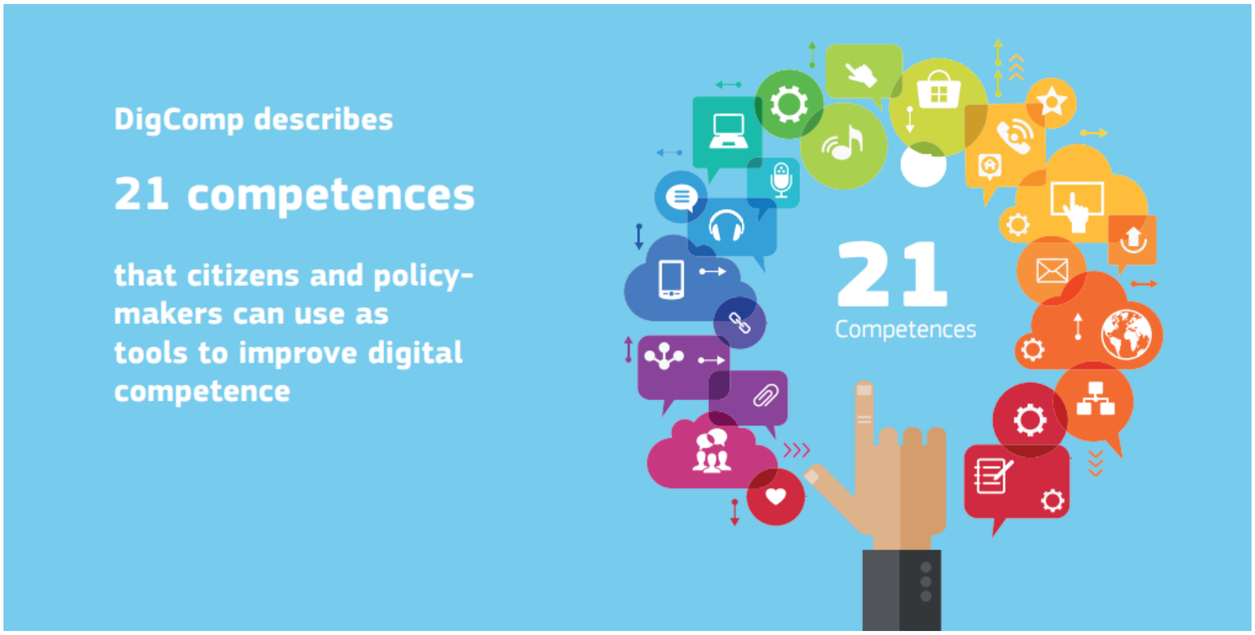
2.2 Assessment of teacher-specific digital competences

Digital competence is essential for education, professional life and active participation in society. In the case of pre-university education, it is important to understand this competence and, equally, to cultivate it.

Digital competence is one of the eight key competences, materializing in the confident and critical use of the entire range of information and communication technologies for information, communication and problem solving in all areas of life. Although it seems simple to many of us, according to the 2015 Digital Agenda scoreboard, the digital literacy of 40% of the EU population is insufficient, with 22% of citizens not using the Internet.

Another aspect worth considering is that "as a transversal competence, digital competence helps us to master other key competences, such as communication, language skills or basic skills in mathematics and science", as Riina Vuorikari pointed out in her article.

To better understand the nature of this competence, the European Commission has designed the European Digital Competence Framework for Citizens (DigComp), structured in five areas: digital and information literacy, communication and collaboration, digital content creation, security and problem solving. The five areas have 21 competencies.



In the near future, teachers will have their own framework, called DigCompEdu. According to the preliminary draft, six areas of development have been defined: the professional environment, the creation and exchange of digital resources, the management of the use of digital tools, the evaluation, the empowerment of students and the facilitation of students' digital competence. We invite you to follow the news about this initiative!

The 2012 Eurydice report shows that almost all European countries have a national strategy for digital competence. At the time of the study, digital competence was taught, through an interdisciplinary approach, in primary education in almost all EU countries (except in two), and in secondary education in all countries, in parallel with other approaches used in some countries, such as integration of ICT in certain disciplines or its teaching as a separate discipline.

One of the key factors is to ensure a sufficient level of digital competence among teachers. According to the latest TALIS study, conducted by the OECD in 2013, 18% of trainers and teachers believe that they need a deeper development of ICT skills for teaching, and 16% related to the use of new technologies in the workplace.

Europe's largest network of teachers, eTwinning, provides teachers with an ideal environment to collaborate with colleagues and discover new ways to use ICT in teaching. The eTwinning study (2015) showed that 29% of teachers consider that eTwinning had a significant impact on their ability to use technologies in teaching, and 37% said that the impact was at least average. eTwinners also reported an increase in the use of digital teaching-learning practices: participation in online courses (78%), creating materials in collaboration with students (77%) or using social networks with students (76%).

Digital skills are systems integrated knowledge, skills, abilities, attitudes and values, formed and developed through learning, which an individual possesses and which can be mobilized to solve various problems that arise in the process of collecting, storing, processing and disseminating information through information and communication technologies.

As technologies are integrated into all activities in any field, the ability to use these technologies and keep up with their rapid evolution has become a condition mandatory because digital technologies are transforming every aspect of life, from lifestyle staff at work.

In the 21st century, teachers need to have digital skills to cope the challenges of the information age. The need to have a certain level of skills digital has two facets for teachers: an integral part of professional skills; incentive and catalyst for the development of evelior's digital skills as a prerogative of their employment.

Areas of competence have been determined in accordance with a number of documents national and international [...], belonging to the field of transversal key digital competence from the framework document of the European Parliament and of the European Council of 18 December 2006 on key competences for lifelong learning []. From the broad spectrum of digital skills have been selected and included in this document those strictly necessary for a teacher to carry out professional activities: access to the best educational resources and content; communication with colleagues, parents and in professional networks; performing the management of teaching activities through digital tools.

Defining the competencies of teachers in the form of a framework document to guide their training and professional development throughout their career is one of the major conditions of increasing the level of student achievement. Digital skills have been classified into three levels - basic, intermediate, advanced - for several reasons.

Currently, at the end of the initial training, teachers have a certain level of skills digital awareness of their curricula and their informal and non-formal education. On theon the other hand, many pedagogues completed their initial education long ago, when many information tools and technologies were not available. At the same time the field of ICT it is a very dynamic one, the speed of development of new tools and opportunities being very great, and the range and complexity of skills needed to be a teacher in the 21st century it is so large that it is unlikely that an individual will fully own or own them develops to the same extent at certain times.

As a result, teachers need to constantly update and master their skills adapt, and this requires critical, evidence-based attitudes that allow them to be responsible for student outcomes.

The skill levels set will serve as indicators in the employment of teachers, as well as in awarding the categories of practicing teachers.

The approach is consistent across European countries in terms of defining competence digital as a key competence. Almost half of European education systems do reference, as regards digital competence, to European definitions of key competences, 11 education systems use exclusively their own national definition of digital competence and eight other countries (Estonia, France, Cyprus, Lithuania, Malta, Austria, Albania and Serbia) use both the European definition as well as the national definition . In general, these definitions are found in curriculum or high-level strategic competence documents.

In secondary education, the number of countries where digital skills are taugh as Compulsory separate discipline increases in more than half of education systems. In the upper secondary education, the number of countries where digital skills are taught as cross-curricular theme declines slightly compared to lower secondary education and there are fewer countries with separate compulsory subjects for all students in this field.

We still have to keep that in mind that in upper secondary education, students can normally choose several subjects optional and these may include disciplines related to digital competence.

Iceland, Greece and Northern Macedonia have the highest number of hours recommended for information and communication technologies (ICT) as a separate compulsory subject in education primary (approx. 150 hours). Lithuania and Cyprus allocate the largest number of hours in education lower secondary, although they did not recommend a number of hours for primary education. In the compulsory education, Romania has the largest number of hours allocated to digital competence as a separate compulsory subject in upper secondary education.

Curriculum reform is currently taking place in half of European education systems digital competence. The review process aims at either introducing competence digital in the curriculum, if not addressed, or to increase the importance of this disciplinary area.

Some reforms also concern changing the curricular approach, updating its contents strengthening certain areas, such as coding, computational thinking or security.

2.2 Development of higher cognitive processes

The modern philosophy of education seeks to focus on a creative procedural, active and cognitive concept of teaching. The innovative approach to education is based on several theses: to arouse enthusiasm in students, enthusiasm for cognition and learning at school, to clearly formulate the goals of student education, to give meaning to student learning - not to memorize the only correct answer, but to construct one's own knowledge (Hanuliaková, J. 2015 Activating teaching, Bratislava: Iris, ISBN 978-80-8153-036-4)

Švec (Švec, Š. 2011. Innovative approaches in didactics. Towards a successful renewal of education. Bratislava: UK. ISBN: 978-80-7097-843-6) understands cognitive education as process-oriented teaching, which is not just about acquiring knowledge, but it also focuses systematically and conceptually on the method of obtaining them. Teaching is focused on activating cognitive functions.

For the needs of teaching work in the school environment, the answer to the question is important: "How should I achieve the set expected educational outcomes?

What means should I use to do this? ”Traditional teaching methods such as interpretation, discussion or practice are not enough to achieve higher educational goals and to move students from obtaining information to the creative process in interaction with the teacher. (Duchovičová, J. - Škoda, J. et al. 2013. Psychodidactic conception of the curricular and mediation context of education. Ústí nad Labem: Jan Evangelista Purkyně University. ISBN 978-80-7414-658-9).

2.2.1 Active learning

Quality textbooks and students' passive approach to teaching are no longer enough to achieve exceptional teaching results. Many educators and researchers today agree that people learn most quickly through their own personal experience and by combining new knowledge with those already known and mastered. Students need to build their knowledge actively and independently, by researching the problem, analyzing it, asking questions, planning research, their own experiments, analyzing their findings and discussing them (Janečková, M. 2015. Discovery teaching of mathematics. Nitra: UKT Nitra. ISBN: 978-80-558-0925-0).

By active learning we mean procedures and processes by which the student receives information with active contribution and on the basis of which he / she draws his / her own conclusions. It processes this information and then integrates it into the system of its knowledge, skills and attitudes. In the form of an active approach to obtaining new information, students also very effectively develop the ability of the so-called critical thinking. This analytical-synthetic process is characterized by its own discovery, assessment and incorporation of new information into the already existing knowledge system, autonomous, individual decisions about their use or rejection.

Based on the experience of teachers and the answers of students Sitná (Sitná, D. 2009. Methods of active teaching: cooperation of students in groups. Prague: Portal. ISBN: 978-80-7367-246-1) states that students themselves prefer activity, cooperation in groups, discussions, debates, they like to work on practical tasks, they like problem-based teaching, they like to solve tasks that require creativity and their own opinion.

Fisher (in Formelová, Z. 2015. The use of innovative methods of teaching mathematics in primary school.) Argues that "any learning that is not entirely routine (and such learning is undoubtedly so) requires creativity. If the student does not know a certain area of study completely, he needs creativity in order to acquire, adapt and apply an understanding that is only partial or incomplete. Accepting new knowledge, developing new ideas or discovering solutions to new problems requires creative thinking and creative, ie divergent thinking offers an opportunity to discover more in every situation than usual. "

2.2.2 Teaching methods developing higher cognitive processes

A well-known criterion for the division of teaching methods is the division of teaching methods according to the nature of the student's cognitive activity and the nature of the teacher's activity, which organizes and manages this diverse activity. Lerner (in Blaško, M. 2016. Quality in the system of modern teaching [online]. [Cited 28.9.2016]. Available at: http: //web.tuke.sk/kip/main.php? Om = 1300 & res = low & menu = 1310) on the basis of it he set aside five cognitive teaching methods, namely interpretive-illustrative (acquisition of ready-made information); reproductive (learning the way of working by imitation); interpretive-problematic (acquisition of reasoned information); heuristic (gaining experience from creative work by staging the problem); research (independent problem detection).

A relatively new and well-known classification is the division of teaching methods according to the criterion of increasing complexity of educational ties, in which Maňák and Švec (2003) distinguish three groups of methods:

- classical methods: verbal, illustrative, practical,
- activating methods: discussion methods, heuristic, situational, staging, didactic games,
- complex methods: frontal teaching, partner teaching, critical thinking, brainstorming, project teaching, open teaching, learning in life situations, computer-assisted teaching, suggestopedia and superlearning, hypnopedics.

Experience with the introduction of innovative activating teaching methods, which can help teachers build a productive atmosphere for classroom learning, make math more attractive and accessible,

is the essence of active teaching (especially its popularity among students) and includes in particular:

- method of discovery,
- heuristic method.

Discovery method

Today's knowledge society needs to respond flexibly to the demands of the times. Students will no longer be satisfied with the knowledge they have acquired through memorization and memory learning. Discovery teaching already suggests from its name that these are:

- discovering, researching the world,
- formulation of questions, discovery of laws, their precise verification,
- leading to new knowledge about phenomena and laws,
- supporting students' curiosity about teaching, creativity and deep experience of their own active work (Hanuliaková, J. 2015. Activating teaching. Bratislava: Iris. ISBN 978-80-8153-036-4).

When the method of discovery is well planned and implemented, it is an active form of learning - a challenging but manageable and fun process. With its help, the student will understand the curriculum very quickly. The method of discovery usually motivates all students, except those completely apathetic. However, it is important to realize that if a teacher thinks badly or misbehaves, students may learn, be confused and frustrated.

For the effective use of the method, it recommends following the following seven main principles:

- students must have all the necessary knowledge and skills that will be necessary to successfully complete the task,
- students need to understand exactly what is required of them,
- the vast majority of pupils must be able to fulfill the task,

- students'

  work must be closely monitored,
- choose a topic where students are unlikely to know the answer in advance,
- students need enough time to research,
- at the end it is necessary to summarize everything that the students had to learn.

During lessons with elements of discovery teaching, students take some control of the teaching process, and the teacher takes on the role of mentor, motivator and facilitator. Its task is to select appropriate situations and problems for students, to clarify the intent of the activities taking place during the lesson, to challenge students by effectively asking questions, to effectively lead discussions in smaller groups and within the whole class. They must constantly support students in exploring and exchanging ideas in a free and reflective atmosphere, helping them to process information and find links between their ideas, and encouraging discussion about alternative methods and understanding. They seek to eliminate the fear of failure by seeing mistakes as an opportunity to learn and not as problems to be avoided. The teacher should also design and use unstructured tasks that are appropriate challenges for students and help students learn in a variety of contexts and situations (Janečková, M. 2015. Discovery of Mathematics. Nitra: UKT Nitra. ISBN: 978-80-558- 0925-0).

The reasons for the implementation of discovery teaching of mathematics are based on the undeniable benefit of the discovery method for students and their learning. Discovery mathematics teaching:

- improves pupils' mathematics performance, with a strong emphasis on less self-confident and disadvantaged students,
- has a positive effect on the attitude and motivation of students, they find mathematics more interesting,
- pupils remember and understand mathematics knowledge faster and easier,
- increases the ability of students to use knowledge in new situations and contexts (knowledge transfer),

- provides

  students with additional skills development opportunities, such as group work, open problem-solving experience and other interdisciplinary skills,
- supports a higher level of intellectual skills and the development of key competences,
- enables students to perceive mathematics, its essence and the way it is created and developed in a more balanced and realistic way,
- Matemat Makes math more accessible to all.

Discovery is recommended to be introduced into the teaching of mathematics as an effective didactic tool for the development of students' knowledge and skills, as discovery leads students to understand the material and to use the existing knowledge and experience. It encourages them to perceive learning as an activity they do and are also responsible for its results.

Heuristic method

Leading student activity to heuristic activity focuses on the active acquisition of general knowledge and skills, the development of creativity, the acquisition of the logic of scientific thinking in stages. Pupil activity is an independent solution of pupils' tasks step by step, in situations that the teacher designs. These are problematic tasks that are differentiated into gradual steps (Hanuliaková, J. 2015. Activating teaching. Bratislava: Iris. ISBN 978-80-8153-036-4).

In order for students to be able to solve a certain problem independently and comprehensively, it is necessary to teach them the individual steps of problem solving:

- analyze the conditions in relation to the issue of the problem,
- to transform the basic problem into a series of partial problems subordinate to the main one,
- design the plan and stages of problem solving,
- formulate hypotheses,
- synthesize different directions of research,
- verify solutions.

In the heuristic method, the teacher manages the research of the problem, formulates opposites,

plans the individual steps of solving the problem, the process of finding out, but the solution of individual steps is performed by the students independently. Students perceive the problem task, think about its conditions, solve part of the task, while updating their current knowledge. They control their solution, justify their actions and actions. At the same time, however, they do not plan their actions, staged surveys (solutions). This is done by the teacher (Blaško, M. 2016. Quality in the system of modern teaching [online]. [Cited 28.9.2016]. Available at: http: //web.tuke.sk/kip/main.php? Om = 1300 & res = low & menu = 1310).

The heuristic method can be implemented in several ways, for example:

If the teacher finds that a certain solution to a problem problem causes students great difficulties, he / she will give them partial, simpler problem tasks, the solution of which is part of the solution of the original task, and after solving them he will return to the original task.

Pupils gradually, step by step, apply the general procedure of solving tasks, in relatively new conditions, for example when solving tasks from a new thematic unit.

The teacher develops a system of tasks, usually in the form of questions, which he arranges so that each subsequent question follows from the previous questions. These questions are small problems and their summary represents a solution to the original problem (Gogolová, D. 2013. Some interactive methods of effective teaching. Bratislava: MPC. ISBN: 987-80-8052-515-6).

The sequence of steps in solving the problem can take place according to Zelina (Zelina, M. 1996. Strategies and methods of child personality development. Bratislava: Iris. ISBN 80-9670-134-7) in the heuristic scheme DITOR: Define the problem. Find out about the problem. A problem solving creature. Evaluate solutions. Implement the solution.

An interesting heuristic method is PHILIPS 66. Its author Donald Philips recommends creating groups of six members (1 leader and 5 members). The problem will be mentioned and the group will discuss it, solving the problem within six minutes. Then the leaders of the individual groups concentrate so that the other members of the group can see and hear them well and report on the results of their group's work, defend them and try to find the optimal solution. The other students are watching them. If no common solutions are found and there is a big difference in the opinions of the individual groups, another round follows, or the teacher summarizes the opinions and draws a conclusion. In addition to practicing creativity, this method teaches students to quickly produce ideas, make decisions, improve their communication skills and abilities (Turek, I.2014. Didactics. Bratislava: Wolters Kluwer, s.r.o. ISBN 978-80-8138-004-5).

P. Zieleniecová (Zieleniecová, P. 2012. Discovery in school - heuristic teaching method [online]. [Cited 28.9.2016]. Available at: https://kdf.mff.cuni.cz/vyuka/pedagogika/dopl_texty/ Heuristicka% 20metoda% 20vyuky.pdf) lists several features of heuristic teaching:

Learning is motivating, fun and students are actively involved in the teaching process. Students are happy that they can solve something on their own, they can do something. They perceive learning as an activity they perform independently.

It leads to a clearer and deeper understanding of the curriculum, to revealing a broader connection with the existing knowledge and with the daily experience of the student.

Supports the development of higher-order thought operations: analysis, synthesis synthesis, and developing creativity. Students learn to identify a problem, formulate questions and hypotheses, solve problems and look for appropriate solutions, design experiments, compare, gather and sort information.

Teaching is divided into larger and smaller steps, in which the student works actively and independently (the length and difficulty of the steps is chosen by the teacher depending on the age and level of students, their previous experience with independent work).

The teacher has ongoing feedback. He is informed about the students' progress and their understanding of the curriculum.

It leads to a deeper and more permanent acquisition of the curriculum.

The heuristic method prevents students' learning from slipping into superficial memorization.

The heuristic method is suitable for the acquisition of the curriculum, which consists of causal relationships, for the acquisition of generalized concepts, relationships, principles, laws, theories and general methods of work, the acquisition of which by independent work exceeds the possibilities of students.

The heuristic method places high demands on the teacher, it is more difficult to prepare compared to the preparation for a traditional lesson. However, it also brings a lot of positive things to teachers. It employs a great deal of his creativity, maintains his teaching motivation, leads him in a natural way to self-education and its results are a source of satisfaction for teachers from pedagogical work.

It is obvious that the complex development of the student's personality undoubtedly requires the inclusion in class of questions, exercises, tasks that will increase the level of all cognitive processes and not only some of them. Without realizing concepts such as memorization, understanding, implementation, analysis, evaluation, or production, even the best teacher can slip into using teaching methods and procedures that are convenient, and teaching can become unilateral, focused solely on memorizing memorized knowledge.

2.3 Support measures for the continued development of teacher-specific digital competences

ICT and Mathematics in the Programme for International Student Assessment

Both in the scientific literature and in the national evaluation reports, it is possible to trace an overview of the impact of the use of ICT at school and student achievement in secondary school with a focus on science subjects such as mathematics, physics and science in general.

The tool used at the European level at the national level is the Programme for International Student Assessment (PISA), better known by its acronym, is an international survey promoted by the OECD created with the aim of evaluating, every three years, the educational level of adolescents in the main industrialized countries. The results reported in this report are from the 2018 PISA analysis which involved 600,000 students representing about 32 million 15-year-olds in schools in the 79 participating countries.

The evaluations performed with this tool allow us to understand whether the institutions that have increased the use of ICT at school have improved educational outcomes more than the others. This allows us to capture the effect of the different policies adopted by schools on the use of ICT. On average across OECD countries, 76% of students attained Level 2 or higher in mathematics.

At a minimum, these students can interpret and recognise, without direct instructions, how a (simple) situation can be represented mathematically (e.g. comparing the total distance across two alternative routes, or converting prices into a different currency). However, in 24 countries and economies, more than 50% of students scored below this level of proficiency. On average across OECD countries, only 2.4% of students are capable of advanced mathematical thinking and reasoning. On average across OECD countries, 78% of students attained Level 2 or higher in science.

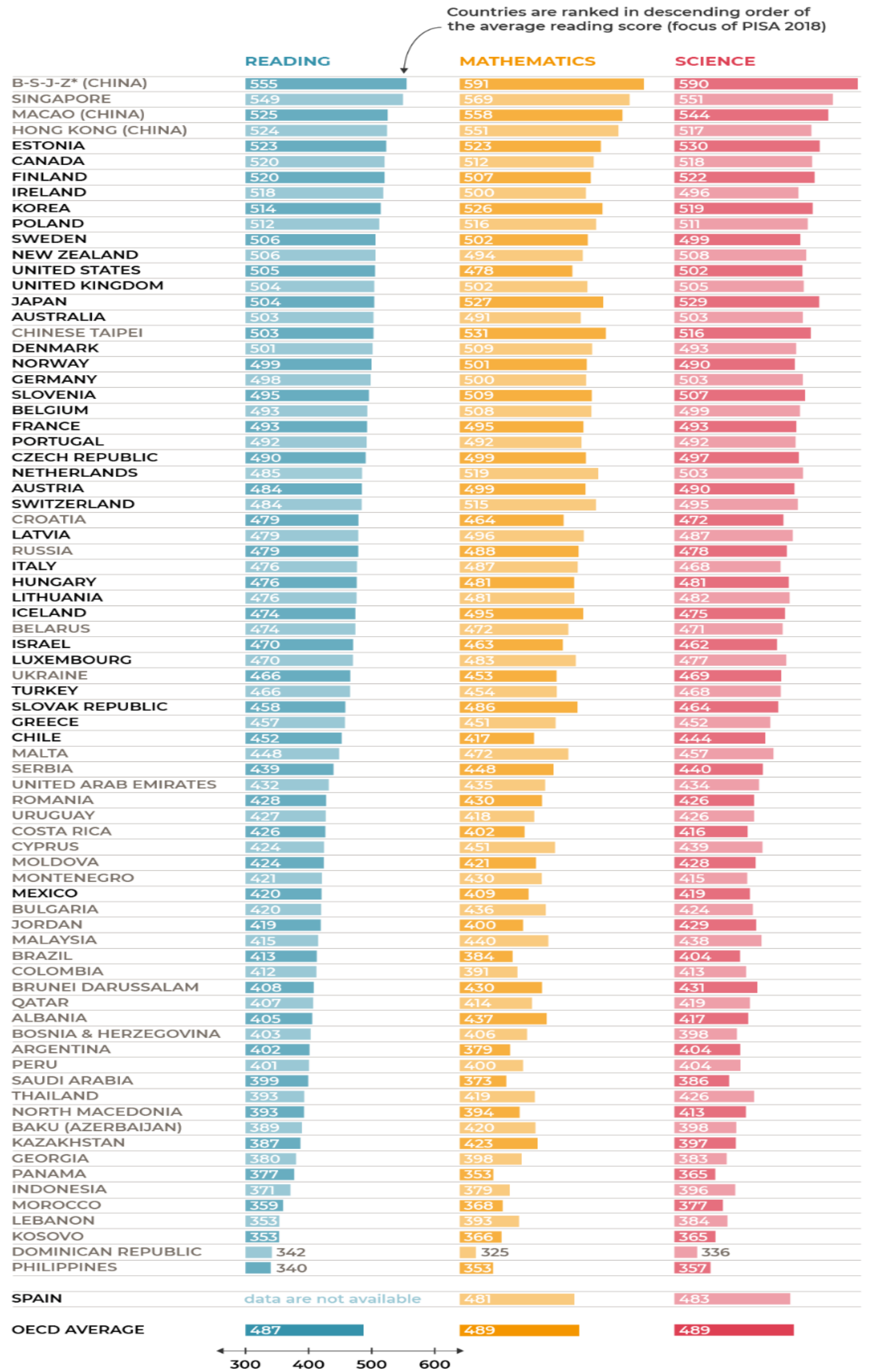
At a minimum, these students can recognise the correct explanation for familiar scientific phenomena and can use such knowledge to identify, in simple cases, whether a conclusion is valid based on the data provided. Regarding the role that digital technology can and should play in schools, the PISA assessment also considers how digital tools have fundamentally transformed the world in and out of school. comparison of PISA data from 2000 to 2012 reveals how the linked introduction of ICT use at the school level had a positive influence on learning outcomes in mathematics and science, and self-confidence in Internet tasks was found to be beneficial for both mathematics and science, and thus suggestions were made to develop students' confidence in performing ICT-related tasks.

In the 2009 PISA assessment about 15 percent of students in OECD countries,on average,reported not having access to the Internet at home. By 2018, this percentage had reduced to less than 5%. The growth in access to online online services is likely to be even more rapid than suggested by these percentages, which mask improvements in the quality of Internet services and the explosion of access to theInternet over the past decade.

Project funded by: Erasmus+ / Key Action 2 - Cooperation for innovation and the exchange of good practices, Strategic Partnerships for school education (European Commission, EACEA)

PISA 2018 results

Snapshot of students' performance in reading, mathematics and science



Source: OECD, PISA 2018 Database || *B-S-J-Z refers to Beijing, Shanghai, Jiangsu and Zhejiang

OECD

Regarding ICT-related variables that may have an effect on learning, two types of use were examined:

- use of programs and software
- use of the Internet.

The frequency of program and software use was measured as an important indicator of ICT use in the years 2000, 2003, and 2006. The data obtained revealed that this type of use had a negative influence on student achievement in both mathematics and science, highlighting that time spent by students using ICT programs and software is not naturally linked to better school performance.

Regarding Internet use, two subtypes of activities were considered:

- Internet for entertainment
- Internet for study

In this case, it was seen that the two modes of use have different levels of influence on students' academic performance. The frequency of Internet use for entertainment, assessed across all five PISA cycles, has a significant impact on achievement in both math and science. From 2000 to 2009, the intensity of students' engagement in Internet entertainment activities had significantly negative relationships with their academic performance.

However, this trend changed in 2012 when data revealed that the influence was positive. The second type of Internet use, study-related, was only measured in 2009 and 2012. Within this activity, a further subdivision was then made based on Internet use at home or at school. In 2009, Internet use for educational purposes had a significantly negative influence on achievement in science subjects, regardless of whether it occurred at school or at home. In 2012, however, analyses revealed that the impact of this type of Internet use was different. Use at school and at home had a positive influence on student performance in math and science.

2.4 Applicability of digitization of mathematics courses with students

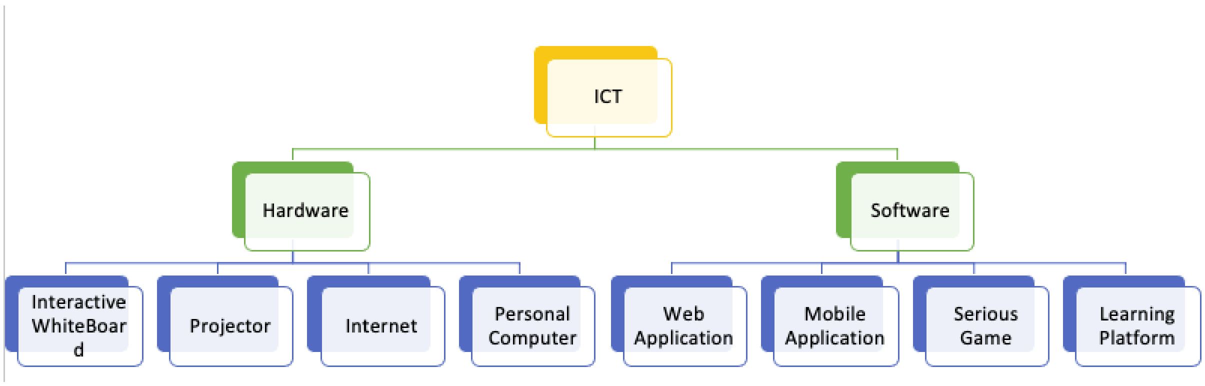
ICT for Teaching and Learning Mathematics

Technology resources dedicated to the study of mathematics allow for opportunities to enhance and stimulate student learning through the possibility of experiencing an active method of problem solving that is both pedagogically sound, technologically innovative, but also interactive and fun.

Normally software for mathematics is used to reinforce the development of specific mathematical techniques; the strategy of use is trial and error, centred on the feedback provided by the system to the action of the student.

The use of technological tools in the classroom, although it may help students to find motivation, it is certainly not enough to ensure the permanence of motivation neither to guarantee the permanence of motivation nor to promote reflective and conscious learning. This is particularly true for new technologies.

In fact they can help improve learning and contribute to the construction of knowledge, they can guide and control interactions between learners, but for this to happen, the learning environment must be appropriately constructed and teaching activities must be carefully designed.



We can classify ICT in 2 big group Hardware and Software. Inside the hardware categories we can include:

- Interactive WhiteBoard
- Projector
- Internet
- Personal Computer
- Electronic Device (General purpose)

In the software categories:

- Web Application
- Mobile Application
- Serious Games
- Learning Platform
- Digital content (General)

2.5. Conclusions from each country

In Romania, the percentage of teachers who have participated in training courses in the last 12 months prior to the study was 89%, below the average percentage in OECD countries participants in TALIS (2018), which was 94%. Instead, a high percentage of teachers reports that they need training in advanced skills in the field information and communication technology (ICT), teaching methods in multicultural environments / multilingual and teaching methods for students with special needs. Both the participation rate and the need for training in these areas has increased over the last five years. From the analysis of the data of the school year 2019-2020, the structure by age groups of teachers (incumbents and substitutes) show us that out of the total number of specialized teachers in education professional and technical (TVET), 27.86% are aged 55 and over.

It is important to anticipate measures to replace specialized teachers for the following profiles / specializations acquired through their initial training: Rail transport (75% are aged 55 and over); Mechanics / Agricultural Mechanics (67.86% are aged 55 and over); Energy / Electric power, thermal energy, hydropower (62.75% are aged 55 and over); Engineering sciences / Mines, oil and gas; Mechanics / Oil and gas (61.5% are 55 years of age and older); Constructions and public works / Constructions; science engineering / civil engineering (60.7% are aged 55 and over); Mechanics / Metallurgy (60.22% are 55 years of age and over);

Ship mechanics (60% are aged 55 and over); Telecomunications; Engineering Sciences / Engineering Electronics and Telecommunications; Electronic and telecommunications engineering; Electronics and automation / Telecommunications (59.46% are aged 55 and over).

For a complete and relevant analysis of the specialized teachers in IPT is it is imperative to complete the existing databases with information on continuing training courses traversed by them. These internships allow IPT teachers to teach disciplines / modules in qualifications that are acquired through IPT, other than those in the regulations on the employment of pre-university teaching staff.

It is also necessary

to redefine the employment of specialized teachers in IPT through distinguishing between their initial training, in correlation with the fields, qualifications and modules of IPT training that they can teach, and ongoing training through training internships that allow the extension of qualifications / training modules that can be taught. New approaches in this direction should allow and stimulate the employment of associated teachers from the turn of specialists in the real economy.

Although there is a national project on the training of teachers' digital skills, implemented by MEC and financed by the Human Capital Operational Program 2014-2020 - "Relevant curriculum, education open to all" (C.R.E.D.), SMIS code 118327 -, aimed at of 55,000 teachers trained (by 2022) to use open educational resources and

for the change of teaching approach, however, there is a major shortage of digital skills professionals at the level of teachers in the pre-university education system in Romania.

These skills are not just about using the computer in teaching, but about a certain type of teaching-learning strategy, about the development of open educational resources, about a certain attitude towards technology and so on.

From the analyzes performed, a series of logistical, pedagogical and technical impediments emergendad content in many school subjects. Among the difficulties in carrying out the activities distance teachers, teachers signal, in order: lack of tools for class management, for feedback and evaluation, technical difficulties - platforms that need installed, which does not work, lack of pedagogical support for learning activities sufficiently effective and / or attractive to all students: lack of the right tools for teaching-learning assessment in their discipline, lack of educational content (digital resources) in the field of discipline, the lack of a sufficiently efficient computer and the lack of time required for understanding and proper use of digital tools and resources.

"Immaturity" of skills to use new technologies in authentic teaching contexts makes it difficult to choose and use dedicated platforms for teaching activities in the environmentonline.

Moreover, the software tools commonly used for synchronous communication with the students were not designed for the purpose of teaching. To this is added a very important element, which ultimately determines the quality of the learning situation: exercise the use of the technological environment for the design and development of complex teaching activities, relevant, meaningful, integrated into long-term learning.

37% of teachers consider themselves advanced users of new technologies (attention, but not necessarily advanced users of educational platforms), and 53% consider themselves intermediate level users.

This image can be a consistent starting point for further activities of continuous training for the development of the digital competences necessary in the didactic activity, with emphasis on virtual classroom, online learning platforms, applications and virtual labs.

The existence of a pressure for digitization through continuous professional training has also been identified teachers. Training courses, webinars, CRED project, CCD activation are needed, especially on the dimension of technology use (special communication applications and platforms and management of learning and assessment processes).

Thus, a very important signal was given for the initial training, we currently have a training and poor digital education, both for the use of the management of current activities (management and process administration), as well as as a tool for managing teaching activities - learning - assessment.

Through the Project "START IN CAREER THROUGH DIDACTIC MASTER" 17, MySmis 140783 will be created Competence profile of the didactic master's degree graduate. It will be structured according to the provisions of the European Framework for Digital Competence of Teachers: DigCompEdu18, intended to support national, regional and local efforts in developing digital competence a educators, providing a common frame of reference with common terminology and logic.

Subsequently, through the project "PROFESSIONALIZATION of teaching career - PROF" is considered increasing the capacity to provide internships in over 100 application schools as well of didactic mentoring activities. Mentoring and the development of professional communities of type laboratory is a solution for easy and practical development of programs training, using good practices.

The objective of the MEC is to expand didactic master's programs as an effective way and modern training / qualification of teachers for preschool, primary and secondary education through advanced university study programs, which aim at the systematic development of competencies their digital.

In order to develop this document, a staff questionnaire was developed and applied applied online from November 30 to December 2, 2020, to which 170 people responded respondents - persons registered from the initial stage on the smart.edu.ro site with interest in the field initial and continuing training.

To the question "What do you think was the biggest challenge for teachers to learn in when to suspend the courses face to face (March 2020)? " used in the consultation questionnaire related to this document, very few of the 170 respondents established a relationship of interdetermination between teacher learning and the learning of their students, considering the following issues:

- maintaining students' passion for learning;
- personal and student time management;
- formative evaluation in the online environment;
- adapting the curriculum to online teaching;

Most of the answers refer to the lack of endowments with technological and communication means, at difficulties in using technology, in the absence of previous learning and practice experiences.

Part of the answers refers to the need for clarity and structure: although there were many resources, even at that time, the lack of a platform, of a somewhat unitary direction of action (even at the level of a small school) led to a multitude of ideas, sometimes applied chaotically, which students and parents, especially those in primary education, the immediate effect being often a slight demotivation, confusion of all parties involved.

The analysis of the answers also shows a poor understanding of the idea of "teacher learning".

Many of those who responded did not associate the difficulties they faced with the lack of some specific skills (ie digital skills associated with the professional role of the framework didactic) and / or the intrinsic motivation to learn, but other aspects, outside their area of competence and liability (no equipment, platform, tools ready to download, etc.) It is necessary that all teachers to develop strong ICT skills. Only in this way can schools benefit from online technology and all digital learning and teaching opportunities and facilities. Testing initial, enabling training on the individual digital skills needs of teachers, is essential.

In general, respondents emphasized the need for a system of "centralized" resources complemented by quality continuing education, focused on adaptation and up-to-date needs.

They also want that through the tools they can access online, the work norm of teachers to be free from repetitive, inefficient activities and in competition with the need for personalization of learning - teaching - assessment for each student, group of students.

This strategy proposes a solution that meets the very different needs of practitioners in education - from needs to "re-consider the teaching-learning-assessment method and techniques", going through simpler needs, learning how to connect to an e-learning platform and to dictate to students or film themselves as they write on the board, until the creation of elaborate teaching strategies which can be used in the context of online education.

The period of the last months imposed for some teachers the transition from the education model "Teacher-centered" - the teacher decides what content he delivers, under what structure, how he evaluates, how it organizes the teaching-learning-assessment process, based on the "student-centered" education model.

One of the directions for transforming education is the individualization of teaching-learning, an effect of the student-centered approach. This message is complemented by the idea of routes individualized educational, for each student. In this regard, the effort required of staff for the individualization and personalization of the didactic act at student level (or group of students with the same learning characteristics) require support in the form of educational resources numerous, varied, quality and available to meet this need for individualization (at the level of school cycle / year of study, discipline, class, student / group of students - typical, with difficulties of learning, with gaps and gaps in previous acquisitions, with special educational needs, with skills high and potentially superior performance).

The perspective of practitioners is that, in the current context, there is a need to capitalize on the opportunity to better operationalize the determination relations between:

i. the competency profile of the teacher, especially the references to digital competences, and the curriculum for initial vocational training (IPF) and continuing vocational training (FPC);

ii. the competency profile of the graduate, for each schooling cycle;

iii. the curriculum associated with the training of skills in the above profiles.

In several consultation contexts, the need for a content platform that was achieved by the contribution of a small number of teachers-authors, for each discipline and year of study. This means significantly questioning the roles of the teacher, who has in the profile of professional skills explicitly formulated the ability to develop and / or adapt resources learning, including curriculum development.

By accessing pre-developed content (Edulib type, Virtual Library) and diversification roles of human resources involved in education, will create an opportunity for time in the agendas teachers and students for "modeling new skills". Solutions developed on the basis of this strategies means, in the short term, a decrease in the allocation of time by teachers in the direction of the development of educational resources and the use of this time in favor of design teaching (including for e-school), associating educational resources with curriculum competencies available, professionally curated, organized and structured in an accessible (user-friendly) way.

To respect the right of every student to a quality education that contributes to the construction character, personal identity and acquisition of skills according to the profile of the graduate, it is necessary for each teacher to agree to go through an audit process of personal skills and

subsequently pursue a plan to improve them through a variety of dedicated programs: training courses, webinars - support, technical training on access and the use of platforms, etc.

The main proposed tool is a learning platform. Education professionals to practice, from the stage of initial training, the responsibility of self-learning in a safe context. On in short, the platform means the context in which they build their own learning journey, as a puzzle that I can permanently remake, depending on the challenges they have in the classroom.

The platform dedicated to teachers needs to put them on the path to improvement periodically in line with global trends.

The platform will allow the support and modeling of processes to improve staff skills teaching at their own pace, in a space of emotional security. The learning relationship between "University professor" and "future student professor" is partially mediated by technology, which which decreases the anxiety associated with the processes of transformation and confrontation of one's own limits learning and adapting to the new

What will we build together?

• A flexible, digitized, adaptable, quality education system, able to respond

challenges and generate change;

• Active citizens, well integrated on the labor market from the perspective of using

technologies digital;

• Sustainable economic growth, based on the trades of the future;

• Digital development opportunities in the field of education and training for a

digital society and a green economy;

• Strengthened the resilience and functional predictability of the education system in the

era digital.

III. 3D Virtual Worlds and Gamification Techniques for Mathematics Training

3.1 Introduction to 3D Virtual Worlds

Virtual reality (VR) involves experiencing things through your computers that actually do not exist. It can be defined as a believable, interactive 3D computer-created world that you can explore so you feel you really are there, both mentally and physically.

Putting it another way, virtual reality is essentially:

1. **Believable**: You really need to feel like you're in your virtual world and to keep believing that, or the *illusion* of virtual reality will disappear.
2. **Interactive**: As you move around, the VR world needs to move with you.
3. **Computer-generated**: It usually involves realistic 3D computer graphics that require computers that are powerful enough to make believable, interactive, alternative worlds that change in real-time as we move around them.
4. **Explorable**: A VR world needs to be big and detailed enough for you to explore.
5. **Immersive**: To be both believable and interactive, VR needs to engage both your body and your mind.

A virtual world or massively multiplayer online world (MMOW) is a computer-based simulated environment in which many users create a personal avatar and simultaneously but independently explore the virtual environment and participate in various activities, as well as communicate with others.

A generally accepted definition of virtual world does not exist, but they do require that the world be persistent; in other words, the world must continue to exist even after a user exits the world, and user-made changes to the world should be preserved. While the interaction with other participants is done in real-time, time consistency is not always maintained in online virtual worlds.

Numerous 3D Virtual Worlds exist, aiming for various purposes, from socialization and leisure, to more formal approaches such as commercial and education oriented.

A virtual world is a digital environment (usually graphical, usually 3D) completely delivered over the Internet or Intranet, where users are represented by Avatars, interact with each other, interact with and affect their environment in a persistent manner, have no more restrictions placed on them than they can expect in the real world, can decide from a wide range of actions, or even inaction, can build and create within the world, without having to master additional tools, can use the world for a wider variety of different purposes".

Many definitions of VW exist in the literature, all of which have the following characteristics:

- Shared space between multiple users
- Graphical user interface – the virtual environment
- Real-time interaction between users
- Interaction with the virtual environment and with digital content within the virtual environment
- Persistence: it is guaranteed that the VW, its objects, as well as user interaction effects will exist even after the user has left the VW
- User communication through text and/or voice
- Networks of people formulate forming social groups
- User avatars: a digital representation controlled by a human in real time to interact within the VW
- Networked computers (servers) that manage all the data

There's clearly more than one approach to building virtual worlds—and more than one flavor of virtual reality. Here are a few of the bigger variations:

1. Fully Immersive

2. Non-immersive
3. Collaborative
4. Web-based
5. Augmented Reality
6. Scientific Visualization

3.2 3D Virtual Worlds Suitable for School Education

Virtual reality is being more and more used in education, allowing the student to find out, to explore and to build his own knowledge. The constant evolution of technology is taking the education to new ways, much more attractive to the students, making possible the use of new tools, taking to an evolution in the teaching process. Virtual Reality takes an important place in this evolution.

Now it's possible to attend a virtual lecture, discussion, exam, or lab in 3D. There are worlds for every young child, teenagers or adults. The aim of these worlds is to encourage people to learn as many things as possible.

Formal learning , normally delivered by trained teachers who are following a specific list of subjects, is part of one of three forms of learning, the others being non-formal and informal learning.Formal learning should not be confused with 'formal learning theory', which, as the Stanford Encyclopedia of Philosophy reminds us, is: "the mathematical embodiment of a normative epistemology ".
Virtual reality and games have the power to make young people focus more on a specific subject and to transform abstract notions in concrete examples.

Virtual World platforms are the tools for creating highly immersive 3D interactive online environments that can be either replicas of existing physical places, or even imaginary places. They can represent places that are impossible to visit in real life due to restrictions such as cost or safety. There exist proprietary and open-source VW platforms. In the following of this section we will describe the most important and popular proprietary and open-source VW platforms.

Second Life:

Second Life (SL) - www.secondlife.com (2003 by Linden Lab) is the most popular VW platform and has the largest active community. It requires an account and subscription to be able to own your own area in the 3D World. Some of the most important features include:

- 3D graphical environment
- Customizable avatars
- Built-in voice and text communication
- Includes a social network, groups, as well as information and object sharing
- Includes in-world virtual currency
- Has an enormous market: users are able to build objects and write scripts and share them in a marketplace.
- Registration and basic usage is free but there is an option of paying a monthly fee for virtual land to build a home and become 'residents' (however, large building projects require the purchase of a large piece of private land - or an island)
- Private land can be accessible only to specific group members
- Many learning organizations from all around the world are augmenting their current curriculum with virtual learning modules of SL, or even conducting classes and educational programs in SL

Active Worlds:

Active Worlds - www.activeworlds.com (1997) is similar to SL in functionality. There is the option to have a basic account for free usage, however paying a small monthly fee allows a unique name, unrestricted access to the VW, avatar customization, object building, access to social networking and communication features such as voice chat, Instant Messaging (IM) and file sharing.

For more control and privacy over environments, private firewall-protected Universes are available usually for enterprise and educational purposes. These Universes are separate worlds from the main world and their cost varies. A separate set of worlds and a community for educational projects with over 80 organizations also exists under the name "Active Worlds Educational Universe".

Jibe:

Jibe is a 3D VW platform developed by ReactionGrid Inc. An important aspect of Jibe is that the developed VWs can be embedded in webpages and accessed from mobile devices. VWs can be either hosted by ReactionGrid Inc., or deployed on private servers. Jibe requires the installation of the Unity web plugin with Android and iOS support under development.

The Unity 3D editor must be used to build a Jibe VW. The outcome is a professional development environment with professional quality graphics, physics and sound. Jibe supports creating 3D objects, as well as importing 3D models from Maya, Blender, and other platforms.

Unity:

Unity – unity3d.com/unity is not a VW platform; rather, it is a 3D professional game development tool which can also be used to create suitable training simulations and educational 3DVWs. It can be accessed through a client or a web based player.

Unity can be used to develop a game along with its user interface without the need to program. The development of single-player games/apps requires only to download and install Unity but the features and properties of the developed training environment depend mostly on the ability to use the content creation tools.

The Unity Asset Store is a Unity global marketplace which provides content such as character models, landscape painting tools, game creating tools, audio effects, visual programming solutions and scripts for free or low cost. Unity evolves with the latest mobile (iOS, Android), desktop (PC, Mac, Linux) and console technology.

Some of the key characteristics of the free version include:

- NVIDIA PhysX Physics Engine
- Audio 3D
- Multiplayer Networking
- Mechanim: complete animation system for characters and objects
- Professional graphics
- Single-Click game deployment to any platform
- Web-browser integration
- Scripting API (supports JavaScript, C# and Boo)

OpenSimulator:

OpenSimulator, Opensim - www.opensimulator.org (2007) is a free, open-source, 3D application server for the development of multiuser 3DVWs. The VWs are accessible through various open source clients. Moreover, VWs can be private or public. OpenSimulator is based on C# and is easily extensible through external modules.

OpenSimulator started as an open source server to Linden Lab's Second Life open-source client. In this manner, the architecture of OpenSimulator is heavily influenced by that of Second Life, allowing the user to produce similar highly detailed 3D graphical environments at low cost, or at no cost provided that the hardware/software and the building/scripting/technical skills are offered for free. Furthermore, the avatars are fully customizable and resemble those of Second Life.

The in-world communication is based on chat and IM. Free voice service with lip sync is currently provided by Vivox Inc., after request. An important feature of OpenSimulator is Hypergrid. Hypergrid is a protocol that allows hyperlinking between Opensim VWs and supports seamless avatar transfers among these worlds.

The aforementioned and the freedom to anyone of owning, building and configuring the VW have made OpenSimulator quite popular among the educational and science community.

Architecture:

- Client – Server
- Client ('viewer'): open-source, multiplatform, free downloadable
- OpenSim Server: open-source, multiplatform, free downloadable

Key Characteristics

- Physics engine: OpenDynamics (ODE)

- Text and IM

communication in-world. Voice not built-in: through external free modules: Vivox, Freeswitch, Mumble

- In-world scripting: Linden Scripting Language (LSL), OpenSim Scripting Language (OSSL), C#

- Built-in 3D building tool
- Hypergrid protocol
- High quality fully customizable avatars
- Image, video, audio file upload (free)
- 3D COLLADA format meshes upload (free)
- Web pages and media on virtual objects
- Region and avatar inventory backup and upload
- Access and building rights restriction
- Groups and full user profiles
- Terrain modification
- Bots (or NPCs)

- Head Up Displays (HUDs)

OpenWonderland:

OpenWonderland, OW - www.openwonderland.org is an open source Java toolkit for creating 3DVWs. The graphics are rather simplistic but other features of the platform are comprehensive. The toolkit allows the creation of modules that can extend the client or the server functionality-wise. Moreover, customized, special-purpose VWs can be created.

Examples of the external modules that have been created by different developers (can be found in Module Warehouse) are: Authentication system, webcam viewer, writable text/HTML poster, collaborative text editor, and more.

A distinct feature of Open Wonderland is easily embedding existing content. There is an enormous list of document types that can be dragged and dropped into the world. Moreover, any content within the Google 3D Warehouse can be imported. Open Wonderland does not offer in-world 3D building; 3D objects can be imported from Maya, Google SketchUp, Blender, etc.

Within OW Vws, users can communicate with high-fidelity, immersive audio, share live desktop applications and collaborate in an education or business context (simulations, meeting rooms, mixed-reality worlds, etc.).

Some ot the key characteristics of OpenWonderland include:

- Scripting in Javascript, PHP, Groovy, JRuby, Java, Jython
- In-world embed scripts in 3D models
- In-world run Open Office, Firefox, NetBeans, etc.

- Shared application framework (whiteboard, PDF viewer, sticky notes, etc.)
- Screen sharing
- Drag-and-drop content
- High quality immersive audio
- Telephone integration
- Webcam viewer, video player, audio recorder
- Group and private text chat, private voice chat
- Portals ("teleport") creation to locations on same server or on different servers
- LDAP plug-in for connecting to existing LDAP authentication systems

Open Cobalt

Open Cobalt, OC - www.opencobalt.net/ is an open source VW browser and toolkit for creating VWs. OC uses peer-to-peer architecture instead of the client-server schema. Peer-to-peer technology enables users to access OC VWs on LANs, intranets, or across the Internet without the need to access remote servers. In this manner anyone is able to host an OC VW for free.

Open Cobalt's peer-to-peer technology through which interactions within VWs are conducted constitutes a great differentiation from other commercial multi-user VW platforms such as Second Life, where in-world interactions are done through central servers. The biggest advantage is that users of OC can set up VWs and interact with other users without any hosting fees, licensing or virtual land lease costs.

OC allows users to hyperlink their VWs and form large distributed networks of interconnected collaboration spaces. It also allows the set-up of public or private 3D virtual workspaces that feature integrated web browsing, voice chat, text chat, and access to remote desktop applications and services.

However, OC lacks 3D content creation tools in-world. It provides the infrastructure for world creation, navigation and collaboration, while it supports content created in open source applications such as Sketchup or Blender.

Key Characteristics of Open Cobalt include:

- "Squeak" for in-world and system scripting
- Navigable 3D hyperlinking between virtual worlds
- Enterprise directory access (LDAP)
- Text chat, in-world voice chat (through VoIP)
- In-world web browsing (via VNC)
- In-world annotations (text and audio)
- Collaborative document sharing/editing
- Save/restore virtual worlds
- Access to remote applications (via VNC)
- Basic end-user content creation and editing
- 3D objects, mesh, texture, media (audio, videos) import
- Avatar selection from custom avatars

3.3 Introduction to Games-Based Learning and Gamification

Games-Based Learning enables new forms of teaching and learning by transforming the traditional instructional processes into playful and interactive story-based scenarios (Pellas & Vosinakis, 2018). As an umbrella term, it has been broadly associated with various definitions emerging from terms like *gamification*, *edutainment*, and *Serious Games*. Nevertheless, despite the interchangeable use of these terms, the conceptual and practical notions that underpin them differ. *Game-Based Learning*, at a simple level, can be defined as "learning that is facilitated by the use of a game" (Whitton, 2014).

Amongst the many definitions that have been coined to define *gamification* the most prominent ones are provided by Huotari & Hamari (2017) and Landers *et al.* (2018) who describe this approach as "a process of enhancing a service with various design techniques that can be used to add game elements to existing real-world processes". *Edutainment* is yet another topic which has been explored from various perspectives with scholars and practitioners (e.g., Charsky, 2010; Colace *et al.*, 2006; de Fossard, 2008) defining it as "a type of entertaining process during which instruction is supported by the aid of multimedia sources (e.g., software, internet, computer games) to exhilarate learners' interest in addition to educate".

Finally, the term *Serious Games* usually refers to "games used for training, advertising, simulation, or education that are designed to run on personal computers or video game consoles" (Susi *et al.*, 2007).

Regardless of the adopted terminology or definition, the core essence of these approaches remains the same: *students learn through repetition, failure, and the accomplishment of goals* (Wu *et al.*, 2011).

Aligned to this notion, Bober (2010) distinguishes Games-Based Learning practices into two main categories: *(1)* learning directly from [playing] the game (constructivist approach) and *(2)* learning from teacher-driven activities related to the game (instructional approach). On these grounds, much discussion has begun to define and classify the main characteristics of digital games as well as their potential in education.

By aggregating and blending information identified in previous review studies (e.g., de Freitas, 2018; Connolly *et al.*, 2012; Wouters *et al.*, 2013), it can be concluded that (educational) digital games need to satisfy the following criteria: *(a)* provide ample opportunities for interaction both between the users and the digital content and the users themselves and *(b)* offer immediate and continuous feedback to the engaged users.

The first criterion is achieved by integrating preset rules and constraints or via predefined objectives and challenges that the players (students) must attain. As for the provided feedback, such practices are usually associated with visual changes made within the environment as a response to users' actions (Prensky, 2003). From another point of view, Herz (1997) classifies the *game genres* in accordance with the following distinct categories: *(1)* action games related to players' interaction with the system, *(2)* adventure games, *(3)* simulation games, *(4)* fighting games, *(5)* puzzle games, *(6)* sports games, and *(7)* strategy games.

3.4 Game-Based Learning in 3D Virtual Worlds

The aforementioned elements and characteristics have attracted researchers' and instructional designers' interest leading them to integrate such solutions in various educational contexts (formal, informal, non-formal) and fields.

A search query performed across different scientific databases related to "3D Educational Virtual Worlds" returned almost 5.500 results of which, 674 discuss implementations and findings emerging from efforts that have integrated Game-Based Learning and gamification scenarios. As a matter of fact, the trend analysis (Fig. 1) illustrates the increasing interest that researchers and educators have toward this direction which is justified after considering the widespread use of 3D Virtual Learning Environments and the added value of such didactic approaches.

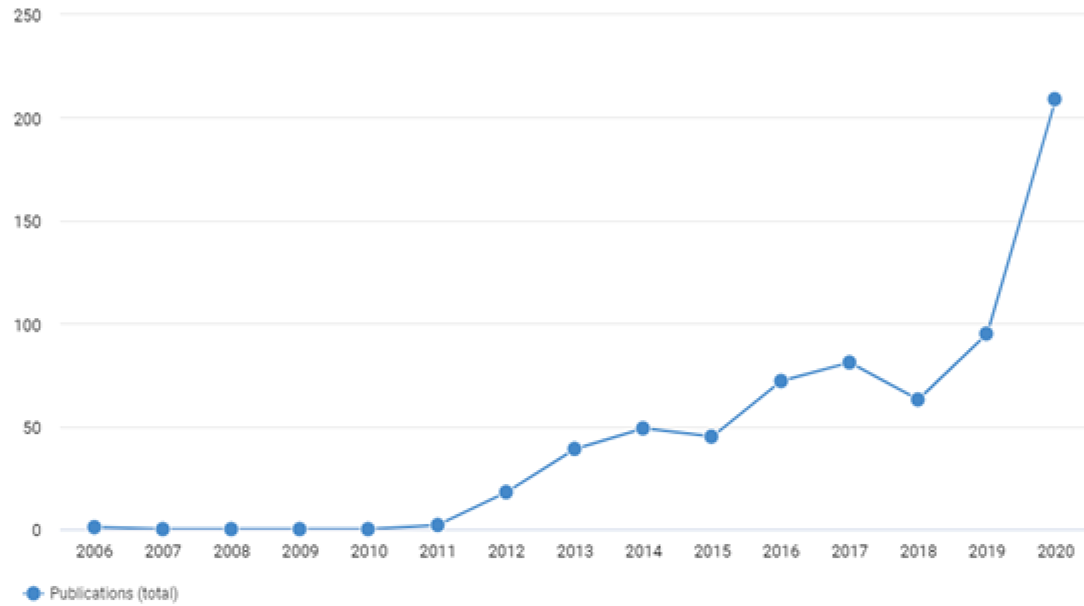


Figure 1. Trend analysis of manuscripts discussing integrations of GBL scenarios in 3D Virtual Worlds.

In most cases, such prototypes allow students to accomplish a number of objectives through activities that involve (Wouters *et al.*, 2013): *(a)* 3D modeling and programming, *(b)* training via simulations, *(c)* experimentation with topics that are difficult or even impossible to be explored in the conventional classroom, *(d)* scenario-based virtual field trips, *(e)* guided explorations via storytelling, and *(f)* assessment of users' skills and competencies in real-time.

Although a portion of researchers considers student engagement in educational games as a self-evident outcome due to the mental and physical effort that students put in achieving their goals, it is not widely known whether the increased degree of engagement is directly related with better learning outcomes or performance.

In view of this consideration, researchers (Connolly *et al.*, 2012; Wouters *et al.*, 2013) recommend that educational games should be first examined from the learnability point of view (i.e., ease of use of the software, difficulty in grasping the game/learning mechanics) and accordingly from their educational perspective (e.g., motivational increase, learning outcomes).

Systematic literature reviews (e.g., de Freitas, 2018; Pellas & Mystakidis, 2020) that have attempted to understand the ways that students learn within such playable contexts indicate that the added value of Game-Based Learning can lead learners to engage more with the subject under investigation and achieve better outcomes or achievements, many times alike or even better than the conventional instruction methods. Other reviews (e.g., Wouters *et al.*, 2013; Connoly *et al.*, 2012) identify connections between learning and engagement which can potentially indicate positive links between the knowledge acquisition and performance.

For example, the findings described in the review conducted by Pellas & Mystakidis (2020) or that of Wouters *et al.* (2013) point out that learning phases and outcomes are associated with student engagement. The same authors also conclude that learners' emotional engagement is also associated with behavioral changes and motivational outcomes, thus indicating anticipation for knowledge acquisition. In the same vein, cognition engagement is associated with knowledge and skills acquisition as well as conceptual understanding (Connolly *et al.*, 2012).

3.5 Integration of Gamification Techniques

The integration of gamified elements in the educational process enables learners to develop their knowledge and skills via playful educational scenarios. To achieve this goal, different *gameplay techniques* are utilised to satisfy learners' preferences, needs, and expectations. According to Salen & Zimmerman (2004), gameplay is the process via which the players interact through a (digital) game. The main venues that connect *gameplay* and *instruction* are summarised as follows:

- *Learning objectives*: Educational games are designed with a specific purpose in mind and clearly defined learning goals aligned to the needs of the learning subject in consideration.
- *Specific instructions and rules*: Educational games have predefined rules and

boundaries so as to prevent the development of misconceptions.

- *Interactivity*: The essence of educational games relies on the active involvement of the learners-players in achieving specific goals and objectives. Therein, diverse opportunities for decisions and actions should be offered during the engagement time.
- *Feedback*: The game should have predefined reinforcement (positive, negative) mechanisms (e.g., rewards, punishment) to compliment the actions of the engaged learners.
- *Challenges*: The integrated tasks should have sufficient degree of scaffolding translated as multivariate difficulty levels. The educational challenges should be proportional to the learners' level and ideally potential.

The *gameplay techniques* are translated to the end users via the integrated *game mechanics* (e.g., game rules, instructions on how to perform specific tasks, challenges that players' face during the playtime) which aim at supporting player's interplay with the virtual environment and other users (Salen & Zimmerman, 2004). Therefore, gameplay in Serious Games relies heavily on the experiential nature of learning wherein knowledge acquisition and skill development are obtained via active participation in the gamified activities which usually involve critical thinking and decision making.

The aforementioned *gameplay elements* can be integrated in different *modalities* in accordance with the requirements that the subject under investigation has. The most widely adopted approaches are presented below:

- *Multiplayer:* Educational games that allow individuals to engage in the learning activities together with their peers (in pairs or small groups) provide additional opportunities for social interaction, boost learners' confidence in achieving greater results, and foster critical thinking.

- *Single-player:*

In individually-oriented games the learners can think of the solution and reflect on their decisions without external pressure from peers. However, to compensate for the lack of social interaction, computer-based agents (puppeteers) provide a great alternative to populate 3D virtual worlds with virtual inhabitants (social characters).

- *Linear events:* Learners are engaged in activities which have predefined steps

and perform a series of tasks that have been organised sequentially. The order of the provided information, guidelines, and virtual tasks should always be consistent with the hypotheses that learners examine each time.

- *Non-linear events:* Learners are provided with multiple data sources, alternate

learning paths, and the freedom to take control over the gamified educational scenario. In this case, learners may decide to skip steps forward or revise procedures that may have not been completed successfully.

- *Context-aware:* Learners' understanding of the actual conditions that govern

the application of similar tasks in the real-world context can be further facilitated with the integration of external to the virtual world activities. In the case of mathematics education, it may involve the use of third-party software or tools that support the visualisation of data.

- *Role-play*: Students can adopt the role of professionals in the context of

problem-solving or simulated activities which include problem deconstruction and analysis, generation of hypotheses, gathering of evidence, and active experimentation.

- *Collaboration:* Working in pairs or small groups with peers enables learners

to discuss their ideas, negotiate problem-solving approaches, and find actual solutions to the provided problems. The sharing of ideas and information facilitates the attainment of

shared goals, promotes the attainment of the learning objectives, and highlights the added-value that collaboration brings in complicated and complex projects.

- *Competition*: In a similar manner to collaboration, competitiveness is also an inherited element that can be found in most modern games. Therefore, providing users with the right incentives to compete with each other (e.g., awards, trophies, tokens) can greatly impact their motivation and facilitate engagement.

3.6 Computer-Assisted Game-Based Learning

Research related to mathematics education demonstrates a supportive stance towards the augmentation of the traditional practices through the use of digital tools (Bussi *et al.*, 2004; Bray & Tangney, 2017). To this end, researchers (e.g., Artigue, 2002; Oates, 2011; Olive *et al.*, 2010), distinguish the added-value of the digital learning approach into two broad categories: *(1)* the *pragmatic*, which regards the efficiency of the tool to facilitate the speed and the accuracy of the computations and *(2)* the *epistemic*, which concerns the knowledge that learners acquire and the ways they develop deep understanding and subject mastery.

Nevertheless, in order for such tools to be effective, special attention should be paid to the design elements of the digital learning environment. Not surprisingly, different studies highlight different variables. According to Rodríguez-Aflecht *et al.* (2018) the technology should be flexible enough, so as to account for the different approaches that learners may utilise towards the solution finding, and capable of facilitating the reflection process upon completion (Heirdsfield, 2011).

Others (e.g., Arnab *et al.*, 2015; Mulligan & Mitchelmore, 2013; Threlfall, 2009; Verschaffel *et al.*, 2009) focus on the practical elements that are responsible for the mental and cognitive engagement of learners (e.g., the utilisation of gamification techniques to integrate the curriculum, the visual representation of the exercises, the escalation of the difficulty level). In any case, researchers and educators share the same opinion as far as the aim of such technologies is concerned by emphasising on the need for change from the *teacher-centered* learning model to the *student-centered* learning approach where learners can control their goals and monitor their progress (Hoyles & Lagrange, 2010; Wright, 2010).

The prerequisite for this change, however, involves the active involvement and continued participation of the teachers for the design of the exercises in accordance with the curriculum changes (Brezovszky *et al.*, 2019).

3.6.1 Mathematics Education in 3D Virtual Worlds

Jabbar & Felicia (2015) classify the key drivers of engagement in the following categories: *(a)* motivational incentives, *(b)* interaction opportunities, and *(c)* fun elements corresponding to usefulness, interactivity, and playfulness, respectively. On these grounds, instructional designers discern the cognitive learning phases in the following sub-categories: *(a)* knowledge acquisition, *(b)* practicing and processing, and *(c)* knowledge application. The identified 3D educational applications related to *mathematics education* utilise various instructional design approaches and gamification techniques.

However, all the gamified experiences feature one common motivational element; they are centered on the various actions or tasks that students can undertake. This not only facilitates the development of higher-order skills but also prevents students from cultivating any misconceptions.

In greater detail, in the first phase, students can actively understand the content by searching and skimming. While processing, they can explore content more deeply and master essential skills that lead to understanding by observing visuals, watching videos, or via intensive reading. In the final phase, students cultivate learning by applying their problem-solving skills (i.e., analysis, synthesis, inference) to prepare the proposed solution.

Some of the most eminent prototypes that have been deployed in schools and training programmes are presented below.

- *Mental calculations*

The prototype developed by Jost and colleagues (2020) aimed at improving students' performance in mental calculation tasks. The educational game can be rendered in either a fully-immersive setups, using a Virtual Reality Head-Mounted Display and haptic input devices, or non-immersive, using portable devices (e.g., tablet computer).

Regardless of the chosen technological aid, the students are 'transferred' to a virtual lake surrounded by mountains wherein arithmetic calculations of variable difficulty are presented at the top of the screen. Underneath the displayed mathematical expressions are located three virtual gates; each one of them displaying a different outcome but only one is correct (Fig. 2).

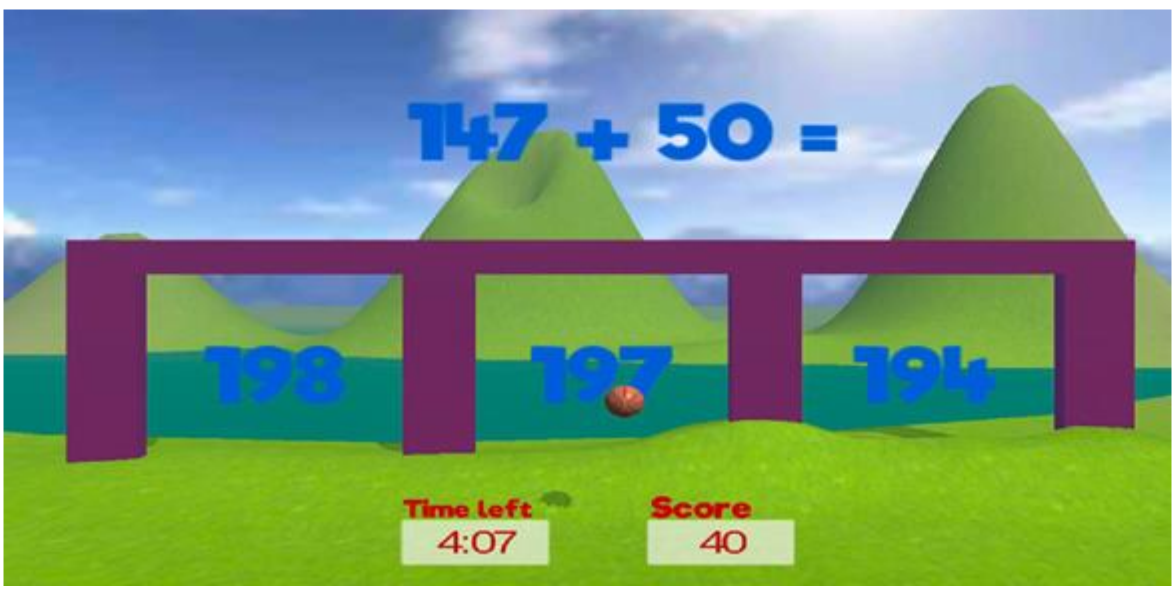


Figure 2. The user interface of the mental arithmetic calculations exercises (Jost et al., 2020).

Students' task is to solve the calculation presented on the screen, using only their mental skills (i.e., without accessing a calculator or any other computational aids). To register their preference over the correct answer the students have to 'throw' a virtual ball over the respective gate. To make the interaction time more enjoyable and pleasant for the students, the designers integrated various acoustic cues and visual effects which are also serving as a means of immediate feedback.

Likewise, in order to increase motivation and engagement, a time counter (5 minutes) and an adaptive reward system were introduced. Students' primary goal was to achieve the highest possible score by answering correctly as many arithmetic tasks as possible. Each correct answer was being awarded by 10 points whereas, for incorrect answers, the students were losing 10 points.

The results of the experimental study revealed that the introduction of gamification elements in exercises related to mental skills' development and mental arithmetic capabilities improvement added an enjoyable note to the learning experience and boosted students' motivation regardless of presentation or interaction-style (i.e., immersive, non-immersive platform setup).

- *Quadratic functions*

Shi and colleagues (2019) designed an immersive virtual reality environment which aimed at *(a)* increasing students' learning motivation and *(b)* improving students' mathematical achievement. The aforementioned points were examined also from the user experience perspective in an effort to identify whether such a solution can become part of the standardised didactic routine.

The educational activities were unfolded in a *virtual park* which included an orientation area and a *virtual basketball court* (Fig. 3). For the needs of this educational activity a special wall with grid lines was readily available for the students accompanied by a *shooting machine.* The game was designed to cover different activities with variable difficulty (4 levels).

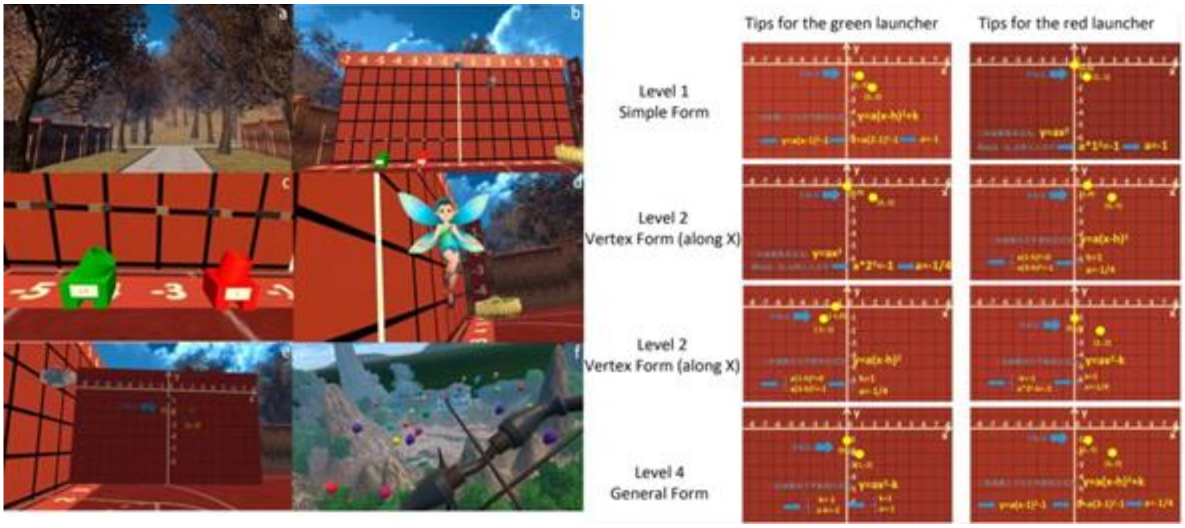


Figure 3. Overview of the virtual park and the math gamified activity (Shi et al., 2019).

To facilitate the learning process students could make use of the *help spirit*, a *fairy-like* help-system (pedagogical agent) capable of redirecting students to short videos which illustrated briefly the computational techniques that required to progress with the given tasks. For every correct solution (*shot*) the students were awarded a *treasure box* which included *gold chips*. Upon completion of the challenges the students were offered the option of riding a virtual crane in the sky.

The experimental study results revealed significant improvements on students' learning performance and enhancement of learning motivation; both confirming the initial hypothesis that 3D-enabled instruction can facilitate complex knowledge learning even when it comes to self–directed learning scenarios. Under this consideration it can be concluded that a well-balanced content–gameplay integration is feasible and can, therefore, be used as part of students' daily activity tasks (traditional class instruction) to turn abstract and complex concepts into student-friendly lessons that are relatable to real world applications.

- *Fractions*

Kim & Fe (2016) created an OpenSim-based virtual world to facilitate students' knowledge development and promote deeper understanding on the topic of fractions. In this learning scenario the activities were unfolded in the context of a *virtual restaurant* (Fig. 4). The rationale behind this design was twofold: *(a)* to offer learners a realistic experience that can be linked and transferred to the real-world setting and *(b)* to assist learners comprehend the subject under investigation by integrating collaborative and adaptive learning techniques.

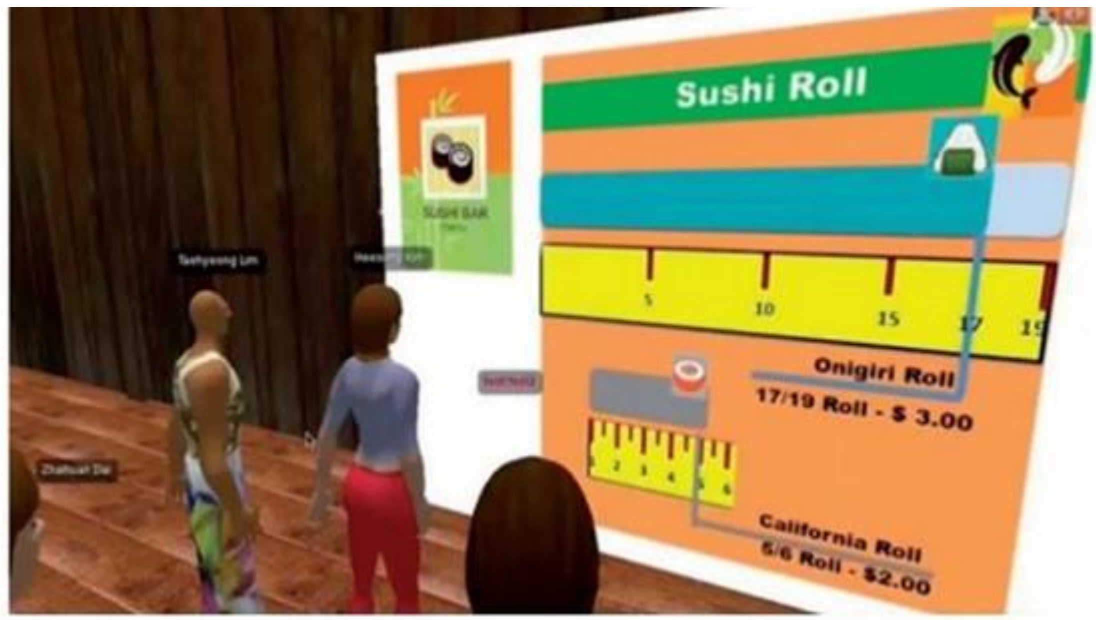


Figure 4. Virtual exercise description in the virtual restaurant (Kim & Ke, 2016).

The key components that contextualised this learning activity included a menu board, providing information related to the subject under investigation, the exercise instructions, and the tasks that students had to perform so as to accomplish them (Fig. 5). In addition, a Non-Player Character, acting as the 'server' (facilitator), was readily available to prompt and assist (e.g., explain the menu options, provide information about special deals) the participating students, when necessary.

The support provided by the pedagogical agent aimed at explaining in a simplified way the concept of fractions as well as the underpinning rules that govern them. Upon successful completion of the activity students were offered a free 'sushi-meal' and discounts acting as rewards of accomplishment.

Figure 5. The gamified menu catalogue (Kim & Ke, 2016).

The findings confirmed that the integration of content-specific graphical representations can assist learners comprehend the subject under investigation in a meaningful and engaging way. Moreover, by enabling them to externalise (i.e., communicate with other students) their mathematical thinking approach facilitated further their problem-solving and computational thinking skills.

Finally, the researchers also emphasise on the important role that the non-human agent played in the educational process as it encouraged learners to discover the solution (via prompts or hints) without providing explicit answers or direct instruction.

- *Coordinate system*

Şimşek (2016) utilised the commercial platform *Second Life®* to design activities that can enable students to understand in a more concrete way the essence of dimensions which are otherwise described in the mathematics course books via 2D model illustrations. For the conduct of this activity a visual illustration of the Cartesian coordinate system was prepared accompanied by a *robot-like* Non-Player Character which facilitated the learning process by providing learners tasks and scaffolding through the game (Fig. 6).

Although no significant differences were observed between the student cohorts (traditional learning versus virtual learning) learning performance, the key-findings revealed that the experimental group (virtual learning) showed an increased interest toward mathematics (asking more elaborate questions) and greater motivation to practice for longer periods of time. The aforementioned outcomes can be further complimented after considering the opportunity given to students to access the 3D virtual world outside the school network which, in turn, brought additional opportunities for social learning and discourse.

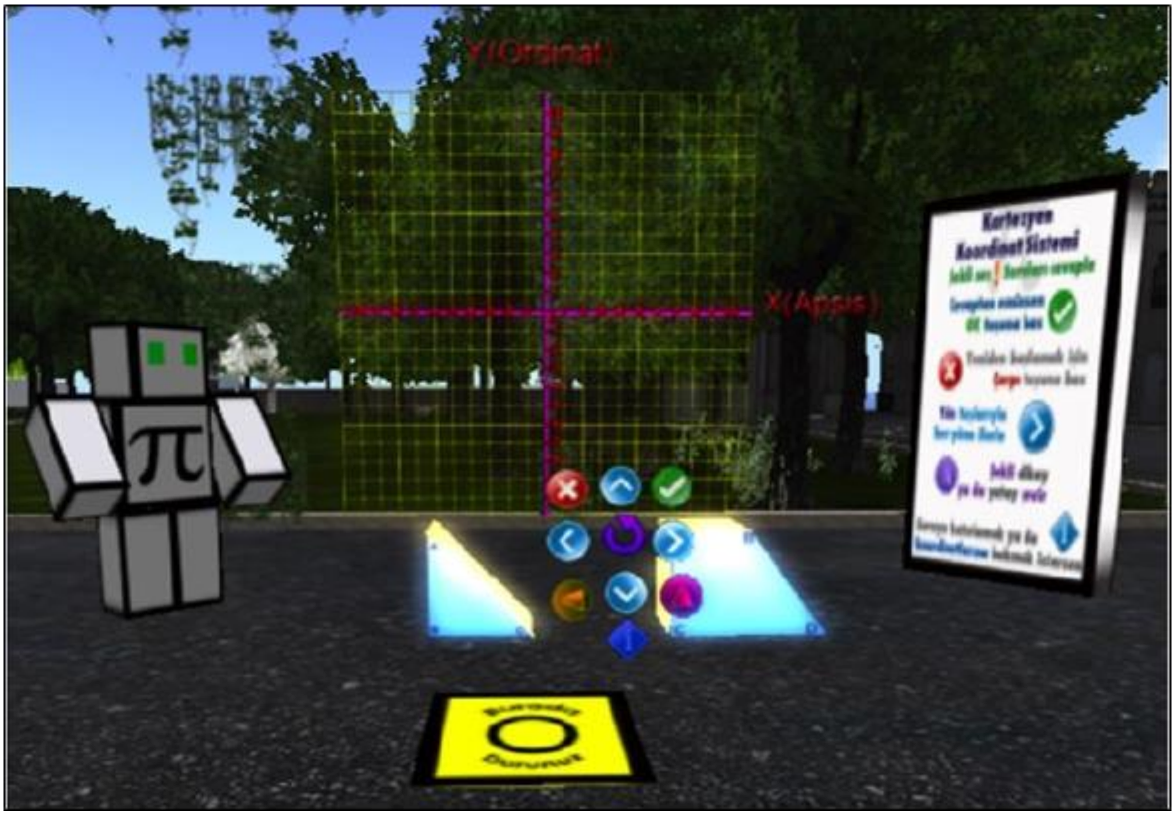


Figure 6. The coordinates system learning scenario and the mathematics robot (Şimşek, 2016).

3.7 Concluding Remarks

3.7.1 Educational Virtual Worlds

For Mathesis Project, there is a requirement for a 3D Virtual World platform that is technically mature, as well as immersive that provides a sense of presence to the users. Immersion refers to both the sense of presence and the quality of graphics, as well as the ability to customize the avatar and interact in an easy, straightforward way.

For the planned learning activities, it is also necessary to support programming capabilities in order to provide interactive objects and special functionality such as Non Playable Characters, according to the scenario requirements.

The most important criteria for the platform selection were:

1. cost-free and open-source
2. support customizable, multi-user Virtual Worlds
3. Good system stability
4. Straightforward server configuration and parameterization in order to fully control the VW and the usage rights at will
5. Self-hosting possibility
6. Reasonable hardware and bandwidth requirements
7. User-friendly, free downloadable, multi-platform client software
8. High-quality 3D graphics and human-like fully customizable avatars to support immersion and sense of presence
9. The platform must be popular regarding educational projects, while a large, active and supportive community of developers should exist

Based on all these criteria, OpenSimulator seems to be the best option for the Mathesis project. OpenSim is a mature platform and fulfils all the aforementioned requirements. In addition, OS's is compatible to SL, the most popular 3D virtual world platform for educators worldwide, as well as its open and modular design, makes the OS platform ideal for educational institutions and enterprises that need to have full control and maximum flexibility on their 3D simulations, while the VWs offer graphics of similar quality to Second Life, similar functionality and similar building possibilities as Second Life but with significantly reduced cost - or no cost at all.

3.7.2 Game-Based Learning & Gamification in 3D Virtual Worlds

Based on the empirical evidences identified in the relevant literature two fundamental instructional strategies were observed in Game-Based Learning contexts: *(a)* game prototypes with lower complexity that relied upon the use of a 3D virtual world to contextualise simulated spaces for observation, exploration, and information collection around predefined tasks (e.g., Merchant, 2010; Tüzün, 2007) and *(b)* game prototypes with higher complexity that provided a richer gaming experience, often with open-ended scenarios, where learners' agency determines the outcome of the game (e.g., Barab *et al.*, Kamarainen *et al.*, 2014).

With regard to the former, such experiences are usually single-player with linear scaffolding (Jakoš & Verber, 2017). On the other hand, the more complex educational scenarios involved the participation of multiple users following a non-linear approach (Barab *et al.*, 2005). Both approaches yielded positive results in terms of students' engagement and learning performance, hence one should not be considered superior over the other.

From the instructional design perspective, many interactive games—such as puzzles, simulations, virtual robots—have been designed and developed in 3D virtual worlds to facilitate the knowledge acquisition process and foster the advancement of skills. When it comes to secondary education settings, a strong preference toward the framing of event-based activities—where challenges and conflicts provide the means to support and enhance students' competencies—is identified (e.g., Jacobson *et al.*, 2015; Şimşek, 2016; Young *et al.*, 2012).

An element that played a crucial role in the instructional process was the use of avatars via which role-playing (e.g., Barab *et al.*, 2012; Şimşek, 2016) and collaborative learning (e.g., Dede *et al.*, 2017; Ketelhut, 2007; Metcalf *et al.*, 2018) activities were facilitated. Likewise, the presence of pedagogical agents (Non-Player Characters), mediated the space between the virtual and the real world or, otherwise, the distance between the students and the instructors (e.g., Jacobson *et al.*, 2015; Nelson, 2006; Şimşek, 2016). Nevertheless, such features were heavily grounded under the notion of narrative-intense and scaffold-oriented storyline scenarios (e.g., Barab *et al.*, 2012; Lim *et al.*, 2006; Young *et al.*, 2012).

Below we provide a summary of the most widely adopted activities/tasks that have been performed in 3D virtual worlds for inspiration and further development:

- Content design and programming, performed by the students, using the available native tools (Pellas & Mystakidis, 2020; Pellas & Peroutseas, 2016; Pellas, 2014; 2017; Rico *et al.*, 2011);
- Visualisation of metaphorical representations and abstract concepts without the constraints or limitations existing in the physical context (Battal & Tokel, 2020; Şimşek, 2016; Young *et al.*, 2012);
- Simulation of problem-solving activities requiring spatial reasoning and perception skills (Pellas, 2014; Yeh & Lan, 2017; Wang *et al.*, 2018);
- Conduct of exploration-based activities in which students need to use computational thinking skills (Jacobson *et al.*, 2015; Lim *et al.*, 2006; Metcalf *et al.*, 2018; Pellas & Vosinakis, 2018; Yeh & Lan, 2017)
- Performance improvement via immediate feedback (visual, acoustic) and opportunities for reflection upon the project completion (Fokides & Chachlaki, 2019; Lim *et al.*, 2006; Mystakidis & Berki, 2018; Zheng *et al.*, 2009);
- Knowledge evaluation and assessment using interactive game prototypes (Barab *et al.*, 2005; Kamarainen *et al.*, 2014; Loula *et al.*, 2014; Metcalf *et al.*, 2018).

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https://pl.padlet.com/features

https://www.glob.org.pl/edmodo/

https://szkolamedialna.pl/edpuzzle-filmy

IV. Examples of Best Practices in national and European level

At the time of the Mathesis project, the entire globe is in the middle of the COVID-19 pandemic. This fact highlights the extreme relevance of implementation of the information and communication technologies, online and blended learning education into each format and level of education.

According to Engelbrecht, J. (2020) educators realise that we need to rethink the entire model of education and redesign it so that it is more student-centred. This means adopting new technologies, but it also means giving up on certain attitudes about what constitutes educational success. These new technologies also seriously influence the nature of mathematics, e.g., application of procedures is becoming less important and new ways of validation (and practising in general) in mathematics are being developed.

The development of new digital technologies provides new opportunities to mathematics educators, and new ways of thinking about how the teaching and the design of teaching–learning environments evolve, generating new practices and establishing goals which we did not think about several years ago.

ICT and innovative technologies tools mediate the actions of mathematics educators, allowing not only to design new learning environments, but this new context also determines new ways of thinking.

In general, there are three different approaches to digital tools (Engelbrecht, J., 2020):

- describing innovative practices (e.g. using digital technology to create virtual instruments with music software or seeking assistance on the internet for help in solving their mathematical problems);
- making explicit principles of design (e.g. the design of assessment tasks for computer based assessment to assess students’ mathematical learning);
- developing a new teaching stage (e.g. innovative design and new organisation of teaching supported by online access to video, shows that the typical lecture and homework elements of a course can be reversed in a flipped classroom

environment).

In the second half of the 20th century, a reform of mathematics teaching called modernization of school mathematics based on its structural-set concept took place in many developed countries of the world, including Czechoslovakia. The purpose of this reform of mathematics teaching was to bring it closer to contemporary mathematics as a science, but for many reasons this "new concept of mathematics teaching" has not proved successful in school practice worldwide.

As stated (Polák, 2020) from the end of the 20th century to the present, a number of theoretical frameworks and practical projects of broader education reforms for the 21st century have emerged worldwide, the purpose and goal of which is to adapt This reform educational effort is coordinated institutionally at the interstate level, inter alia, within the agenda of the Organization for Economic Co-operation and Development (OECD).

Education in Europe is one of the priority areas for transnational cooperation and competences are defined as a combination of knowledge, skills and attitudes. Key competencies include 8 competencies:

- literacy competence,
- competence in the field of multilingualism,
- mathematical competence and competence in the field of natural sciences, technology and engineering,
- digital competence,
- personal, social and learning competences,
- civic competence,
- business competence,
- competence in the field of cultural awareness and expression.

Mathematical competence is the ability to develop and use mathematical thinking and insight to solve various problems in everyday situations. The starting point is a reliable mastery of basic arithmetic operations and the emphasis is on process and activity, as well as knowledge. Mathematical competence includes, to varying degrees, the ability and willingness to use mathematical ways of thinking and expressing (formulas, models, figures, graphs and diagrams).

Mathematical literacy is the ability of an individual to know and understand the role that mathematics plays in the world, to make well-founded judgments, and to penetrate mathematics to meet his or her needs as a creative, interested, and thoughtful citizen.
From the course and results of all curricular reforms of mathematics teaching
It is clear that for its real improvement and adaptation to the requirements of the 21st century, it proves necessary to systematically and convincingly acquaint not only students but also the public about the current practical importance of mathematics as a science and the main goals of quality mathematics teaching. In terms of the binding Recommendations of the Council of the EU, it is possible to define as the main goals of mathematics teaching in the 21st century in particular (Polák, 2020):

1. Development of mathematical and logical thinking of pupils.
2. Acquisition of basic (curricularly determined) mathematical knowledge and knowledge with proper understanding, including the ability of mathematical expression (use of the language of mathematics).
3. Creating mathematical skills of students and their ability to use creatively (applications).
4. Educational influence on pupils through teaching mathematics.

4. Curriculum

In the **Czech Republic**, some elements of EU education policy were implemented in the school system in the first decade of the 21st century, when a major curricular reform took place. The basic idea became the requirement that instead of just discussing the prescribed curriculum, key competencies of students be created and cultivated.
The educational area of Mathematics and its application is based primarily on active activities that are typical for working with mathematical objects and for using mathematics in real situations. It provides the knowledge and skills needed in practical life, and thus enables the acquisition of mathematical literacy.
Education emphasizes a thorough understanding of basic thought processes and concepts of mathematics and their interrelationships. Pupils gradually learn some concepts,

algorithms, terminology, symbolism and ways of their use.

The educational content of the educational field of Mathematics and its applications is divided into four thematic areas.

1. In the **Numbers and Numerical Operations** topic, which is followed and deepened by the Numbers and Variables topic, students learn arithmetic operations in their three components: the ability to perform operations, algorithmic understanding (why the operation is performed by the presented procedure) and semantic understanding (to be able to link the operation to the real situation). They learn to obtain numerical data by measuring, estimating, calculating and rounding. They get acquainted with the concept of variable and its role in the mathematization of real situations.
2. In the topic **Dependencies, relationships and working with data**, students recognize certain types of changes and dependencies that are a manifestation of common real-world phenomena and get acquainted with their representations. They are aware of the changes and dependencies of known phenomena, they come to understand that change can be both growth and decline, and that change can also have zero value. Pupils analyze these changes and dependencies from tables, diagrams and graphs, in simple cases construct them and express them by mathematical prescription or, if possible, model them using suitable computer software or graphic calculators. Exploring these dependencies tends to understand the concept of function.
3. In the thematic area of **Geometry** in the plane and in space, students identify and represent geometric shapes and geometrically model real situations, look for similarities and differences of shapes that occur all around us, realize the mutual positions of objects in the plane (or in space), learn compare, estimate, measure length, size of angle, circumference and content (resp. surface and volume), improve your graphic expression. Exploring shape and space leads students to solve positional and metric problems and problems that arise from common life situations.

4. An important part of mathematics education are **Non-standard application tasks and problems**, the solution of which may be largely independent of the knowledge and skills of school mathematics, but in which it is necessary to apply logical thinking. These tasks should permeate all thematic areas throughout basic education. Pupils learn to solve problem situations and problems from everyday life, to understand and analyze the problem, to classify data and conditions, to make situational sketches, to solve optimization problems. Solving logical problems, the difficulty of which depends on the level of pupils' intellectual maturity, strengthens the pupil's consciousness in their own ability to think logically and can catch even those pupils who are less successful in mathematics.

Pupils learn to use computer resources (especially calculators, suitable computer software, certain types of tutorials) and use some other tools, which allows access to mathematics and students who have shortcomings in numerical computing and drawing techniques. They also improve in independent and critical work with sources of information.

4.1 Pedagogical approaches

4.2 Resources and materials

Umime To

The Portal "Umimeto" (for mathematics www.umimematiku.cz) except the mahematics provides supporting educational content for Czech, English, German languages, programming and geography. In a unified environment, it offers coverage of a wide range of subjects from 1st grade to graduation. Adults will also find interesting exercises here. It offers many different forms of practice. One topic can be practiced in a number of different ways. This helps to consolidate knowledge and provides room for choice according to one's own preferences.

The environment is playful and motivating and offers a number of playful elements. The main emphasis is not on superficial elements of games (graphic and sound effects), but on deeper game principles such as clear and achievable goals, immediate feedback or imaginative assignments. How do you rate it?

The basic principle is the practice system, not testing. The method of evaluation used takes into account mistakes, which is an important part of learning. Every mistake can be corrected by further practice and even a faulty beginner has the opportunity to reach the highest shield.

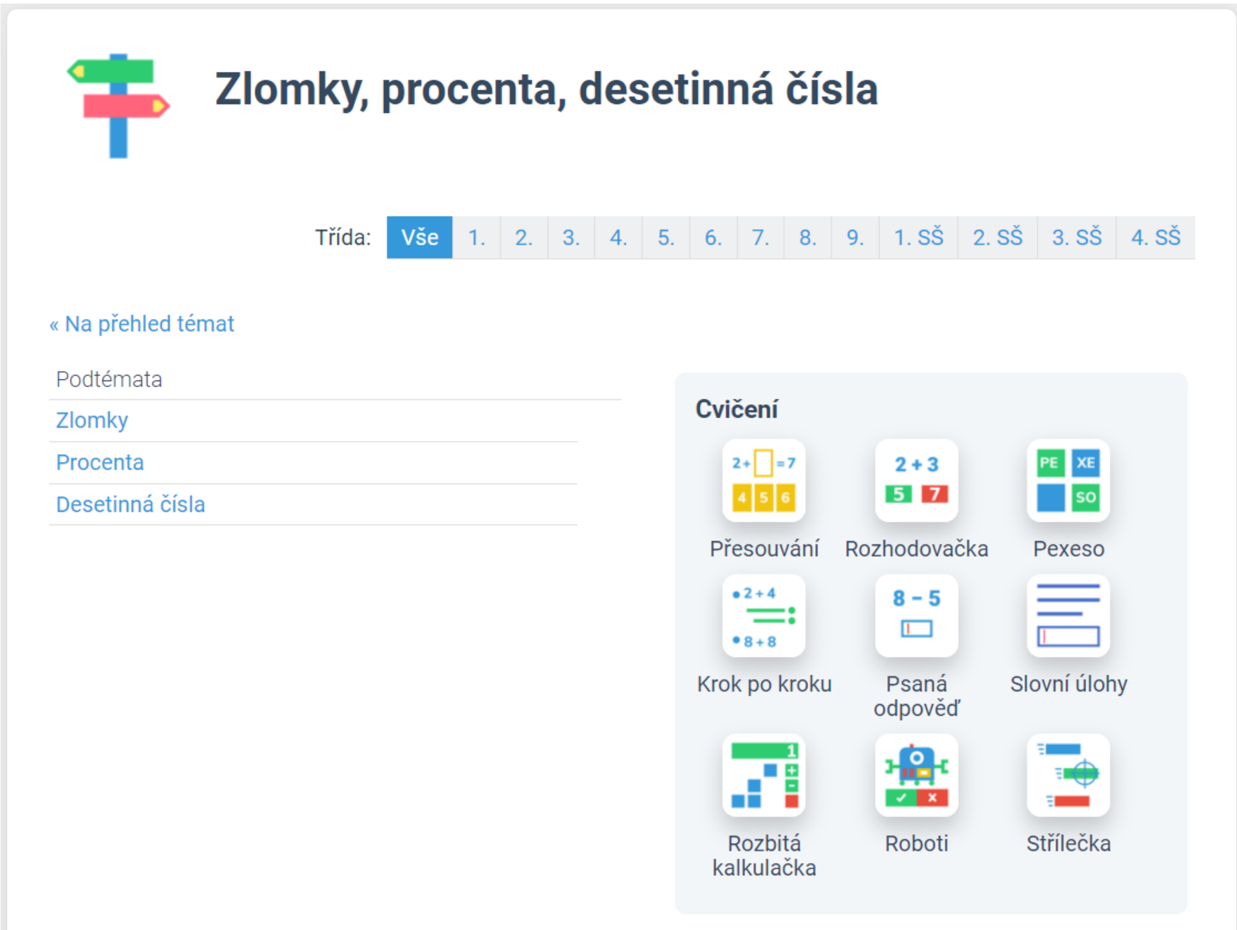

Figure 7. Example of the activitie (Fractions, percentage, decimals) at the www.umimematiku.cz.

Matika je In

Matika je in (www.matika.in) is an online environment focused on training mathematics tasks for children in the first stage of primary schools. it is based on the following principles:

- mathematical environments according to hejný's method: buses, snakes, neighbors, exhibition grounds, grandpa leson, cobwebs, dart chart, triangles, asterisk, indian multiplication, dice…
- mathematical problems with one solution, with more solutions, without solutions.
- working with an error: notification of an error in the solution, the child looks for the error itself.
- reasonable challenges: individual approach, possibility to choose difficulty and tasks

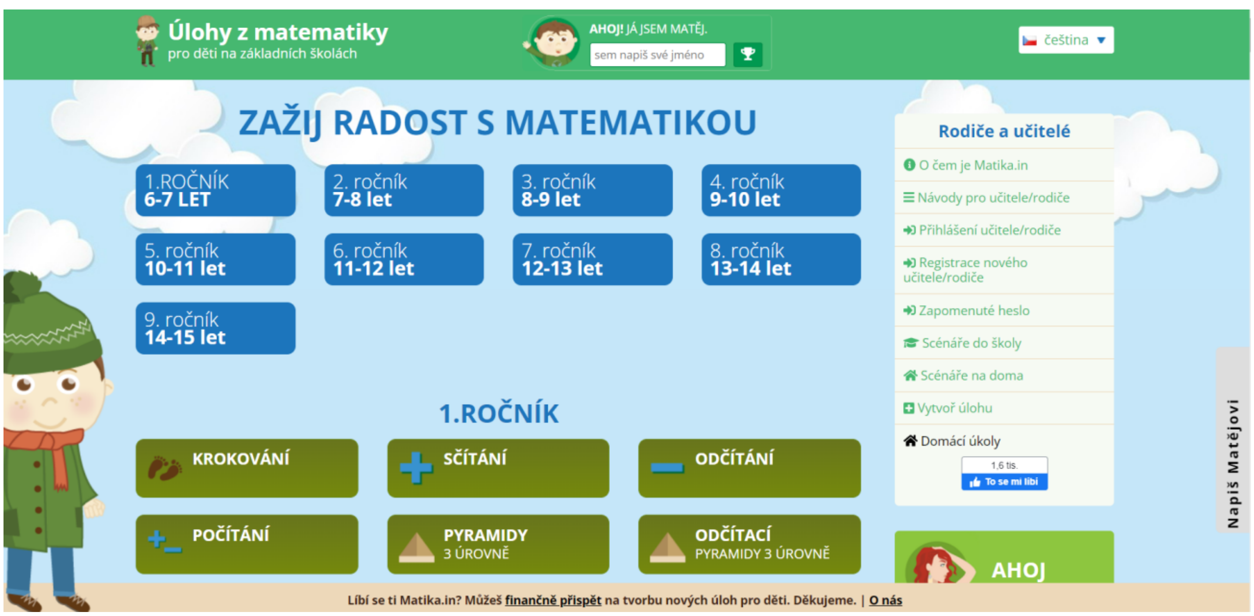

Figure 8. Homepage of Matika je In Portal (www.matika.in)

Matematics at Czech TV

During the COVID-19 pandemic, Cesta televize also joined the support of distance education in the Czech Republic and prepared educational videos, which can be found at https://edu.ceskatelevize.cz/predmet/matematika

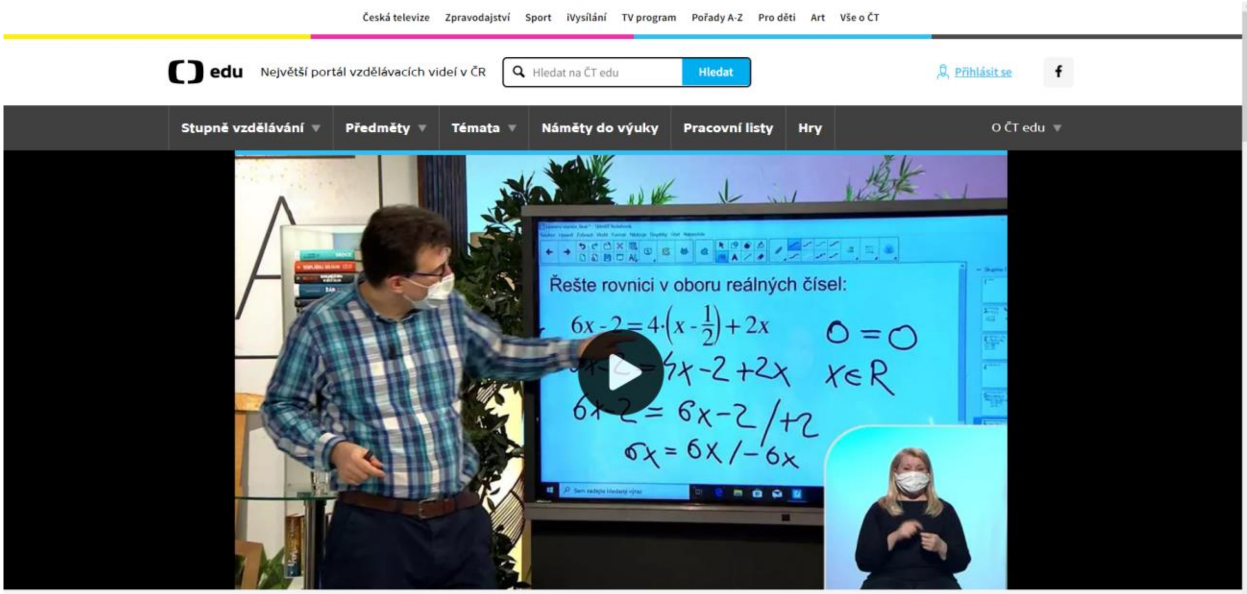

Figure 9. Mathematics at Czech TV

4.3 Professional development and support for teachers

4.4 Experience and educational level

Analysis of experience and practice of use ICT in teaching mathematics

Jones and Tanner (2002) conducted a study within a school in South Wales in the UK that included four secondary schools and eight teachers. In this study, they qualitatively investigated the impact of introducing an interactive pedagogical model based on the use of IWB. Analyses conducted by the researchers during live lesson observations, evidence from group meetings conducted, and interviews with teachers, however, revealed that in practice the adoption of IWB and the strategies adopted by teachers varied significantly from each other. The results of the analysis showed that the quality of interaction varied between teachers and was dependent on the opportunities created within the lesson for reflection and the quality of the questions asked. Thus, it can be deduced that opportunities for reflection and the quality of questions need to be carefully designed to ensure the proper integration of IWB in mathematics education.

Diwakar et al, (2015) analyzed the role of virtual labs (simulated labs that adhere to real labs) in supporting teaching and learning. The virtual lab was integrated within a blended course that alternated online activities and classroom presence.

Questionnaires were administered to the students to assess their perception of the usefulness and adaptability of the tool as a means of self-learning. The analysis of the questionnaires showed how much the students appreciated the virtual laboratories and how much they increased their level of self-efficacy and confidence in these tools. The possibility of being able to access the labs, even if simulated, at any time without the risk of making mistakes led the students to become more involved.

Subsequently, the views of teachers were assessed by administering questionnaires during workshops. The analysis of the results showed that teachers consider virtual labs to be an effective tool for both training and assessing students through experiential activities. The use of such technology also bypasses the problems of access to the premises and the procurement of raw materials needed for laboratory experiments. Ninety-five percent of the teachers interviewed were in favor of integrating this new technology and the blended learning model into their course.

Some case studies are proposed in Bramald's (2000) work. The first case study looks at the use of simple drawing software to support the development of counting skills in 4-5-year-old children. Teachers invited children to use a simple drawing program to create pictures to count. They then counted the created pictures aloud with them to fix the numbers in their minds. Some of the activities involved the teachers presenting some physical objects that were then to be reproduced and counted by the children in the virtual environment.

The children were divided into 2 groups. One group was given the task using ICT and the other group was used as a control group. The analysis of the results regarding counting skills showed that the group that used the technological tools performed better than the control group. This analysis suggests that careful planning of classroom activities using ICT can play a role in improving the counting skills of young learners.

Another case study concerns the teaching of real numbers to 4th-grade students using the concept of time. The activity used an e-Mate, a computer specifically suited for use by children because of its robustness, connected to a pressure-mat sensor.

The activity was carried out by developing two different tasks. In the first task, the young students were asked to climb onto the pressure mat and jump, noting the time they spent in the air. The students took turns in the exercise and the times were stored on a leaderboard, which triggered a competitive environment in which, to the nearest hundredth of a second, the student who managed to obtain the time corresponding to the largest decimal number was the winner. In this informal and fun way, they were introduced to the meaning of decimal numbers. The second activity had the same objective, but in this case, the students were asked to touch the mat, run from one side of the room to the other, and touch up the mat. This activity also stimulated the competitive aspect but in this case, the winner was the one with the lowest number. One participating teacher reported:

> "I found decimal fractions quite difficult for a lot of children. It may be the fact that I didn't expect them to take it much further that was part of my problem.
>
> much further that was part of my problem as well. The e-Mate stimulated their interest to learn and understand beyond my expectations. I would never in my wildest dreams have thought they would cope with this. "

Finally, by testing the skills acquired, it was possible to verify that the students who were trained using this innovative methodology performed better than their peers who studied using traditional methods.

Umbara et al. (2019), demonstrated the efficiency of CAI computer-assisted instruction through the use of Hippo Animator (audio-video animation software) for learning mathematical representations. The analysis demonstrated the effectiveness of the system developed, which is based on a self-regulated learning model in which the student can plan his course of study.

In addition to a significant increase in mathematical representation ability, the students developed a positive attitude towards learning mathematics. This confirms previous studies showing that the integration of ICT into a learning pathway improves positive attitudes (Chen et al.2015)

The effectiveness of the model is determined by the use of multimedia supports such as graphics, images, audio, video, and animation. The description of mathematical concepts supported by multimedia increases the ability of students to focus on learning the concepts presented.

4.5 Teachers attitudes and influence of the environment

4.6. Conclusions from each country

V. Final Conclusions

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